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On the fluctuations in consumption and market returns in the presence of labor and human capital: An equilibrium analysis

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Abstract

We examine the effects of human capital on consumption, stock market, and other fluctuations in a general equilibrium continuous-time model. A representative consumer–worker–investor derives utility from consumption and leisure. A representative firm demands labor as the sole input to a stochastic production technology, driven by general (possibly nonmultiplicative) shocks. For Cobb–Douglas utility and multiplicative shocks, labor is nonstochastic, and consumption and stock market volatility are equated, as under no human capital. Deviations from this are analyzed. For logarithmic utility and 'constant elasticity of substitution' production technology, cases are identified where the presence of labor causes consumption to be smoother than the stock market.

JEL classification: C60; D51; D91; E20; G12

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1. Introduction

The benchmark model in dynamic asset pricing theory is the consumption-based capital asset pricing model, growing out of the work of Merton (1973), Lucas (1978) (pure exchange), and Breeden (1979), Prescott and Mehra (1980), Brock (1982), and Donaldson and Mehra (1984) (production). Such a model implies a very strong mapping between the dynamic behavior of aggregate consumption and stock market wealth. In its most standard form under a

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pure-exchange general equilibrium environment, uncertainty enters the economy entirely through exogenously specified stock dividends, and aggregate consumption absorbs all the dividend shocks. Since aggregate wealth (the present value of future consumption) coincides with stock market wealth, all financial shocks are absorbed completely by aggregate wealth. As a special case, under assumptions most often made in financial economics (logarithmic utility, or constant relative risk aversion preferences and geometric random walk dividends), aggregate consumption volatility coincides with the financial risk in the economy.

It is by now well known that the standard model performs poorly in describing the empirical data on consumption and stock market fluctuations. Time series studies have revealed aggregate consumption to be too smooth relative to market returns to support the claim that the stock market is the present value of future consumption. This discrepancy is also related to the 'equity premium puzzle' (Mehra and Prescott, 1985): the observed excess return on risky assets is higher than predicted given the smoothness of aggregate consumption.[1]

One major element missing from the standard asset pricing models is labor or human capital, the present value of future labor income. It has long been recognized (Mayers, 1973) that human capital constitutes an important part of an individual's wealth, and hence of aggregate wealth in the economy. Stocks, on the other hand, form only a small part of aggregate wealth. As Jagannathan and Wang (1994) point out, the monthly per capita income in the US (during 1959–1992) from dividends is less than 3% of that from all sources, whereas income from salaries and wages is about 63%. There is significant overlap in the segments of the economy that earn this labor income and that invest in the stock market. For example, Blume and Zeldes (1994) find from the 1989 Survey of Consumer Finances that fully 1/3 of all US households own stock, and this population earns 55% of the total income in the economy. A variety of observations of financial investment behavior indicate that households allow their human capital to influence their investment decisions. Examples include (Blume and Zeldes, 1994): households with unemployed heads are less likely to hold stock; as the education level of the head increases, a household is more likely to own stock; and the percentage invested in stocks displays an inverted 'U' shape with respect to investors' age. These data point to the need both to model an average investor's optimization problem as a joint labor/leisure/consumption/portfolio decision, and to incorporate such investors into a dynamic capital asset pricing model. In a model with human capital, total wealth no longer necessarily absorbs all the financial shocks in the economy and consumption no longer absorbs all dividend shocks. Bodie et al. (1992) argue that with labor flexibility, much of the financial uncertainty may be absorbed into labor/leisure behavior, leaving consumption relatively smooth.

[1] A variety of papers extending the standard model to better explain the smooth consumption in data include those of Black (1990) and Constantinides (1990).

Our objective is to develop a tractable continuous-time, general equilibrium model of a representative consumer–worker–investor that combines the labor and human capital aspects of macroeconomic environments with a financial environment. Part of this objective is to investigate theoretically how the interaction between an investor's labor, consumption, and portfolio choices affects the dynamic behavior of consumption and labor supply, financial wealth, and human capital. Our strategy is to deviate in only one dimension – the incorporation of labor – from the standard continuous-time asset pricing models, such as Duffie and Huang (1985), Duffie (1986), Huang (1987), Duffie and Zame (1989) and Karatzas et al. (1990). The subject of labor is, of course, prevalent in the stochastic growth models of the business cycle literature, but there labor choice is typically treated independently of financial investment decisions. Nevertheless, analytical results in these growth models are limited, mostly restricted to the first-order conditions that the equilibrium must obey; as Campbell (1994) points out, 'Despite the wide popularity of the stochastic growth model, there is no generally agreed procedure for solving it ... In [almost all cases] some approximate solution method is required'. The main contribution of our work is the model's tractability, allowing exact and analytical results for all economic fluctuations. Of course, tractability rarely comes without some cost; in this case, we abstract away from capital as an input to the production and from possibly unhedgeable human capital.

Our economy contains a representative consumer who derives time-separable utility from both consumption and leisure, and who simultaneously invests in a stock and bond market and earns a labor income. The consumer's labor is demanded by a representative firm as the sole input to a stochastic non-constant-returns-to-scale production technology. The objective of this firm is to maximize its lifetime profits, paid out to its shareholders as dividends. All the (possibly non-Markovian) uncertainty in the economy is generated by production shocks, which are allowed to appear quite generally, possibly non-multiplicatively, in the production function. Deviating from the assumption of multiplicative production shocks is something of a novelty in our model; the business cycle literature focuses on long-term growth and cyclical behavior, and so for stability must restrict production uncertainty to appear as a simple multiplicative factor. This assumption is unnecessarily restrictive, though, for our short-term fluctuations containing no long-term growth component. Our main tool of analysis is the martingale representation technology (Cox and Huang, 1989; Harrison and Kreps, 1979; Karatzas et al., 1987). All quantities are restricted to constitute a rational expectations equilibrium.

Our general equilibrium analysis allows labor, and consequently all other endogenous quantities, to be implicitly determined in terms of the model primitives: the consumer's preferences and the production technology. The response of labor (and hence consumption, wages, and dividends) to a shock is driven by the difference between the consumer's elasticity of substitution of leisure for consumption, and the firm's elasticity of marginal product with respect to output in response to a shock. When this difference is zero, labor is deterministic, but still important since it drives a wedge between the wage, consumption, and dividend volatilities. Under deterministic labor, the elasticity of substitution drives the relative volatilities of wages and consumption: when the elasticity is higher than 1, wages are more volatile; when less than 1, wages are less volatile. We show a limitation of the most standard case considered in models with labor, Cobb–Douglas utility and multiplicative shocks. We demonstrate this to be the knife-edge case predicting nonstochastic labor and no effect of human capital on the dynamics. Wage, dividend, and consumption volatilities are equated, as is stock market volatility under geometric Brownian motion shocks. Accordingly, we deviate in two directions to induce stochastic labor. First, for logarithmic utility we generalize to nonmultiplicative production shocks, captured by a production technology exhibiting 'constant elasticity of substitution' (CES) between labor and the shock. Second, for multiplicative shocks we deviate to a general CES utility function.

We identify the relative responsiveness to a shock of human capital and total wealth as the important factor in determining whether total wealth is smoothed or made more volatile relative to the standard model. For logarithmic utility and geometric Brownian motion shocks, this reduces to the relative responsiveness to a shock of labor and consumption. In the logarithmic utility and CES production function example, cases are identified in which stock market is more volatile than consumption: when the shock impacts productivity less than it impacts output, or when a 'good' output shock impacts productivity negatively. Endogenous conditions for the reverse to arise are also derived. In the multiplicative production shock example we deduce a deviation from the standard consumption-based CAPM, which may act to increase or decrease equity risk premia. Cases are identified for which the deviation mitigates the equity risk premium puzzle: when substitutability of leisure for consumption and intertemporal substitution are both high but the latter is higher, or the exact reverse. We also weigh our model's implications against pertinent macroeconomic findings, such as: that dividends have the most volatile growth rates followed by labor, consumption, then wages;[2] and that consumption and labor comove positively

[2] This ranking combines quarterly macroeconomic (labor, consumption, wage) data reported by Cooley and Prescott (1995), from 1954–1991, detrended using a Hodrick–Prescott filter, and annual financial data (consumption, dividends) of Cecchetti et al. (1990), from 1871–1985. Although Plosser (1989) reports a reversed ranking of consumption and wage volatilities, based on matching the parameter values of an assumed 'correctly specified' model to annual data from 1954–1985, his methodology is less commonly adopted. Weighing our model against the data is a back-of-envelope comparison, since our model predicts instantaneous volatilities while the available data are over finite periods. Computation and ranking of volatilities over a finite period would require knowledge of conditional variances over that finite period, unavailable except for a few special cases such as geometric Brownian motion, typically not arising in our equilibrium. Under geometric Brownian motion, our results on instantaneous volatilities extend exactly to volatilities over a finite-time period.

(e.g., Blanchard and Fisher, 1989).[3] None of our cases predict every aspect of the financial and macroeconomic data, but several are consistent with most or all aspects. Under logarithmic preferences and CES production function, all our unambiguous conclusions are consistent with the data, except that consumption and labor may covary negatively. Under CES utility and multiplicative shocks, for high substitutability of leisure for consumption and some space of production functions, the unambiguous results are all consistent with the data.

The value of our general equilibrium analysis is highlighted by its use to evaluate several commonly made conjectures regarding the effects of a labor/leisure choice. We focus on three points in particular. A common notion, put forward by Bodie et al. (1992) is that consumption is smoothed if consumers absorb much of the financial shocks into their labor choice. As discussed above, however, we show that the labor choice may instead be used to amplify the financial shocks absorbed into consumption, particularly when labor and consumption comove positively as is observed (Propositions 1 and 7(c)). A related argument made by Blanchard and Fisher (1989) is that time-separable preferences predict a negative comovement of consumption and labor. Since this result is deduced assuming fixed wages, it should be re-evaluated in a general equilibrium context with endogenous wages. We indeed show (Propositions 7(a),(b) and 9(a)) that labor and consumption may also comove positively when the wage response dominates the output response to a production shock. The third common conjecture, the foundation for the work of Hansen (1985) and Rogerson (1988), is that a high intertemporal substitutability of leisure in agents' preferences goes hand in hand with highly volatile labor (relative to wages). This argument is made at a partial equilibrium level, ignoring fluctuations in the valuation of labor's output. By endogenously pricing the market for output, our model identifies a weakness in the argument and shows that labor can be less volatile than wages even under infinite leisure substitutability (Remark 1).

To our knowledge, our model is the first general equilibrium model in continuous time to include both labor supply and dynamic consumption/portfolio decisions. In continuous-time finance, work on production economies has primarily consisted of models such as Cox et al. (1985), where the consumers invest capital directly in stochastic constant-returns-to-scale (CRS) production technologies; the issue of labor is not addressed. Bodie et al. (1992) are the first to tackle the issue of labor choice in a standard complete markets continuous-time finance framework. Their work is at a partial equilibrium level, addressing how the labor decision affects an agent's dynamic consumption/portfolio choice, and vice versa. Our general equilibrium analysis is most closely related to the discrete-time models of Prescott and Mehra (1980), Brock (1982), Danthine and

[3] We focus on this particular comovement because it is the one most commonly referred to and is shown to be a critical factor in the comparison of consumption and market returns volatility.

Donaldson (1994) and Rouwenhorst (1995).[4] Prescott and Mehra, and then Brock extend the Lucas (1978) asset pricing methodology to a production economy. Both Prescott and Mehra (employing a CRS technology) and Brock (non-CRS technology with – 100% depreciable – capital as the only input) show the existence of a Pareto optimal rational expectations equilibrium, but focus little on characterization. Danthine and Donaldson (1994) and Rouwenhorst (1995) adapt a one-sector (also complete markets) version of Brock's model to incorporate labor as a further input to the technology. Their analytical results are restricted to the ensuing first-order conditions. Under the assumption of multiplicative shocks, Markovian uncertainty, and Cobb–Douglas (Rouwenhorst) or log-linear (Danthine and Donaldson) preferences, they calibrate their model parameters to data and solve the model numerically. A related paper is by Danthine et al. (1992), who numerically evaluate the extent of resolution of the equity premium puzzle under several business cycle models; in particular a non-Walrasian formulation with labor contracts provides a partial resolution. These three papers are able to explain some but not all regularities in the macroeconomic and financial data simultaneously. Finally, Mankiw et al. (1985) and Eichenbaum et al. (1988) empirically study (partial equilibrium) representative agent models of consumption and leisure. They estimate the first-order conditions under time-separable CES (Mankiw et al., 1985) or non-time-separable Cobb–Douglas (Eichenbaum et al., 1988) preferences and find that the observed comovements of consumption, labor, and wages reject these first-order conditions. By not being analytical, the authors admit that they can do little to identify the missing features of existing models.

Section 2 of this paper solves the consumption/labor/portfolio problem of the representative consumer and points to methods of comparing financial wealth, total wealth, and human capital volatilities. Section 3 solves the firm's problem. Section 4 defines equilibrium and solves for the pertinent equilibrium dynamics. Section 5 considers the case where nonstochastic labor arises in equilibrium. Section 6 deviates to examples with stochastic labor. The Appendix provides all proofs.

2. The economic setting for a consumer with labor supply

This section presents a continuous-time economy from the viewpoint of a representative consumer–worker faced simultaneously with a dynamic consumption–labor–portfolio choice.

[4] See also Donaldson and Mehra (1984) who adapt a one-good version of Prescott and Mehra (1980) with particular focus on the impact of the underlying economic environment on the market risk premium, and the recent OLG-based asset pricing models in life cycle economies of Constantinides et al. (1997) and Storesletten et al. (1998).

2.1. Information structure and securities

We consider a finite horizon $[0,T]$ economy with a single consumption good (the numeraire). The uncertainty is represented by a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathscr{P})$ on which is defined a one-dimensional Brownian motion $z(t)$, $t \in [0,T]$. All stochastic processes to appear in the sequel are assumed adapted to $\{\mathscr{F}_t; t \in [0,T]\}$, the augmentation by null sets of the filtration generated by z. All stated (in)equalities involving random variables hold $\mathscr{P}$-almost surely. We assume all processes and expectations are well-defined, without explicitly stating the required regularity conditions. Notation σ_X denotes the volatility (instantaneous covariability of the percentage change with changes in the Brownian motion) of an Itô process X.[5]

The financial investment opportunities are represented by two securities, an instantaneously riskless bond in zero net supply, and a risky stock (the equity market) in constant net supply of 1 that pays out dividends at rate $\delta(t)$. δ is endogenously determined via the firm's optimization problem (Section 3). The bond price, B, and (ex-dividend) stock price, S, are assumed to follow

$$\mathrm{d}B(t) = B(t)\, r(t)\, \mathrm{d}t, \tag{1}$$

$$\mathrm{d}S(t) + \delta(t)\, \mathrm{d}t = S(t)[\mu(t)\, \mathrm{d}t + \sigma(t)\, \mathrm{d}z(t)], \tag{2}$$

with $S(T) = 0$. The interest rate, r, and the stock's drift, μ, and volatility, σ, are (possibly path-dependent) adapted processes, with σ bounded away from zero. The security price system (r, S) is to be determined endogenously in equilibrium, with S verified to follow the posited Itô process (2). Dynamic market completeness allows the construction of a unique system of Arrow–Debreu securities consistent with no arbitrage. Accordingly we define the *state price density process* ξ as a process with dynamics

$$\mathrm{d}\xi(t) = -\xi(t)[r(t)\, \mathrm{d}t + \theta(t)\, \mathrm{d}z(t)], \tag{3}$$

where θ is the *market price of risk process*, defined by $\theta(t) \equiv (\mu(t) - r(t))/\sigma(t)$. $\xi(t, \omega)$ is interpreted as the Arrow–Debreu price per unit probability $\mathscr{P}$ of a unit of consumption in state $\omega \in \Omega$ at time t.

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[5] In particular, using the standard shorthand notation, X satisfies $\mathrm{d}X(t)/X(t) = \mu_X(t)\, \mathrm{d}t + \sigma_X(t)\, \mathrm{d}z(t)$, with the property that the instantaneous conditional covariances and variances are given by $\mathrm{cov}_t(\mathrm{d}X(t)/X(t), \mathrm{d}z(t)) = \sigma_X(t)\, \mathrm{d}t$, $\mathrm{var}_t(\mathrm{d}X(t)/X(t)) = \sigma_X(t)^2\, \mathrm{d}t$ so that $|\sigma_X(t)|$ represents the instantaneous standard deviation of percentage changes in X. For two Itô processes X, Y, we have $\mathrm{cov}_t(\mathrm{d}X(t)/X(t), \mathrm{d}Y(t)/Y(t)) = \sigma_X(t)\sigma_Y(t)\, \mathrm{d}t$. In this paper, we will frequently use the ratio $\sigma_X(t)/\sigma_Y(t)$ to quantify comovements of X and Y; this captures the sign of the covariance and also the relative magnitude of variability in X versus in Y.

2.2. The consumer's endowments, preferences, and optimization problem

A representative consumer–worker is endowed at time 0 with 1 share of the stock, and at each time t with $\bar{\ell}$ units of available labor, to allocate between leisure $h(t)$, and labor $\ell(t)$, for which he is paid a wage at rate $w(t)$. The consumer intertemporally chooses a nonnegative consumption process c, labor process ℓ, and portfolio process π, where $\pi(t)$ denotes the amount invested at time t in the risky asset, so as to maximize his lifetime utility. Denoting F as the consumer's financial wealth, $F(t) - \pi(t)$ is the amount invested in the bond, and F must satisfy the dynamic budget constraint

$$\mathrm{d}F(t) = F(t)\,r(t)\,\mathrm{d}t - (c(t) - w(t)\,\ell(t))\,\mathrm{d}t + \pi(t)\,(\mu(t) - r(t))\,\mathrm{d}t$$
$$+ \pi(t)\,\sigma(t)\,\mathrm{d}z(t), \quad F(0) = S(0). \tag{4}$$

The consumer derives a von Neumann–Morgenstern time-additive, state-independent utility $u(c(t), h(t))$ from consumption and leisure in $[0, T]$. The function u is assumed to be homothetic, twice continuously differentiable in both arguments, strictly increasing and strictly quasi-concave in the two arguments, and to satisfy $\lim_{c\to 0} u_c(c,h) = \infty$, $\lim_{c\to\infty} u_c(c,h) = 0$, $\lim_{h\to 0} u_h(c,h) = \infty$, $\lim_{h\to\infty} u_h(c,h) = 0$. This form of utility function allows incorporation of the recent notion of the indivisibility of labor (Hansen, 1985; Rogerson, 1988; see Remark 1), but cannot incorporate the notion of non-time-separable preferences over labor (Kydland and Prescott, 1982).[6] We sometimes further assume the utility function to be in the constant elasticity of substitution (CES) family

$$u(c,h) = \frac{1}{\gamma}[\beta c^{\rho} + (1-\beta)h^{\rho}]^{\gamma/\rho}, \quad \gamma < 1, \ \rho < 1, \ \beta \in (0,1),$$

with elasticity of substitution of leisure for consumption $e_{c,h} = 1/(1-\rho)$, and intertemporal elasticity of substitution (of composite good) $1/(1-\gamma)$. This family includes the familiar Cobb–Douglas utility function, $u(c,h) = [c^{\beta} h^{1-\beta}]^{\gamma}/\gamma$ ($\rho = 0$), and more particularly the logarithmic function, $u(c,h) = \beta \log c + (1-\beta)\log h$ ($\rho = 0$, $\gamma = 0$).

It is well known (Cox and Huang 1989; Karatzas et al., 1987) that using the martingale representation approach, the dynamic optimization problem of a consumer facing a complete market can be converted into a static variational problem with a single budget constraint. Since all processes in our model are driven only by z, human capital, although not directly marketed, is spanned by the available securities, and hence the consumer–worker faces a dynamically

[6] One may generalize our analysis to include non-time-separability by adapting the recent methodologies in 'habit formation' (e.g., Constantinides, 1990), but this is beyond the scope of this paper.

complete market. Standard techniques imply that the consumer–worker solves

$$\max_{c,\ell} \mathrm{E}\left[\int_0^T u(c(t), \bar{\ell} - \ell(t))\,\mathrm{d}t\right] \tag{5}$$

$$\text{subject to} \quad \mathrm{E}\left[\int_0^T \xi(t)\,c(t)\,\mathrm{d}t\right] - \mathrm{E}\left[\int_0^T \xi(t)\,w(t)\,\ell(t)\,\mathrm{d}t\right] \le \xi(0)\,F(0)\,. \tag{6}$$

We do not explicitly apply the nonnegativity constraints $c(t) \ge 0, \ell(t) \le \bar{\ell}$, $\ell(t) \ge 0$, because the conditions $\lim_{c\to 0} u_c(c,h) = \infty$ and $\lim_{h\to 0} u_h(c,h) = \infty$ guarantee $c(t) > 0$, $\ell(t) < \bar{\ell}$, while (in the equilibrium provided) the firm's production technology (Section 3) guarantees $\ell(t) > 0$. The consumer's static budget constraint (6) states that the total cost of future consumption net of labor income must be no greater than the consumer's financial endowment.

The first-order conditions of the static problem (5)–(6) are

$$u_c(\hat{c}(t), \bar{\ell} - \hat{\ell}(t)) = y\xi(t), \tag{7}$$

$$u_h(\hat{c}(t), \bar{\ell} - \hat{\ell}(t)) = y\xi(t)\,w(t), \tag{8}$$

where the Lagrangian multiplier y satisfies Eq. (6) with equality at the optimal $\hat{c}, \hat{\ell}$. Consequently,

$$\frac{u_h(\hat{c}(t), \bar{\ell} - \hat{\ell}(t))}{u_c(\hat{c}(t), \bar{\ell} - \hat{\ell}(t))} = w(t)\,. \tag{9}$$

The consumer's optimally invested financial wealth is given by

$$\hat{F}(t) = \frac{1}{\xi(t)}\mathrm{E}\left[\int_t^T \xi(s)\,\hat{c}(s)\,\mathrm{d}s \mid \mathscr{F}_t\right] - \frac{1}{\xi(t)}\mathrm{E}\left[\int_t^T \xi(s)\,w(s)\,\hat{\ell}(s)\,\mathrm{d}s \mid \mathscr{F}_t\right], \tag{10}$$

the present value of his future net consumption $c - w\ell$. Since

$$L(t) \equiv \frac{1}{\xi(t)}\mathrm{E}\left[\int_t^T \xi(s)\,w(s)\,\hat{\ell}(s)\,\mathrm{d}s \mid \mathscr{F}_t\right]$$

represents the present value of the agent's future labor income or the value of his optimal human capital, the term

$$W(t) \equiv \frac{1}{\xi(t)}\mathrm{E}\left[\int_t^T \xi(s)\,\hat{c}(s)\,\mathrm{d}s \mid \mathscr{F}_t\right]$$

is interpreted as total wealth, so that *total wealth = financial wealth + human capital.* This contrasts with the standard consumption-based asset pricing models with no labor income, where financial wealth equals total wealth and hence coincides with the present value of future consumption.

2.3. Financial wealth, equity market, human capital dynamics

One of our goals is to compare the fluctuations of the stock market, total wealth, and consumption to address the extent to which the presence of labor can help explain the excess stock market volatility over aggregate consumption volatility. To clarify the impact of labor, we consider this question upfront from a partial equilibrium perspective, while anticipating two general equilibrium notions: considering the consumer of Section 2.2 as the representative consumer–worker for the whole economy; and equating the representative agent's equilibrium financial wealth to the equity market price. Hence, Eq. (10) yields

$$S(t) = W(t) - L(t), \tag{11}$$

and so by Itô's lemma, total wealth volatility is given by

$$\sigma_W(t) = \left(\frac{S(t)}{W(t)}\right)\sigma(t) + \left(\frac{L(t)}{W(t)}\right)\sigma_L(t), \tag{12}$$

a weighted average of the volatilities of stock price and human capital.[7] In contrast to a model with no human capital, total wealth indeed no longer absorbs all the financial shocks in the economy. Bodie et al. (1992) conjecture that total wealth absorbs less shock, arguing that consumers use their human capital to 'cushion' themselves, which suggests a negative covariability between human capital and the equity market ($\sigma_L \sigma < 0$ or $\sigma_L/\sigma < 0$). They reason that labor flexibility may induce smoother consumption because consumers can work harder in 'bad times', which suggests a negative comovement of consumption and labor ($\sigma_\ell/\sigma_c < 0$). However, many authors (Black, 1987; Baxter and Jermann, 1993; Campbell, 1996) argue on the contrary that the stock market and human capital are highly positively correlated ($\sigma_L/\sigma > 0$), while it is well known that observed consumption and labor move in the same direction ($\sigma_\ell/\sigma_c > 0$). This would suggest that the extent of absorption of financial shocks by total wealth or consumption is in fact underestimated in a model with no labor. Here, we formalize the notion of using human capital as a cushion, by analyzing the connection between an excess stock market variability and the comovements of human capital and stock market, or labor and consumption. Only our general equilibrium analysis (Sections 4–6), however, can identify environments which lead to a positive or negative relation between labor and consumption.

From Eq. (12) we deduce

$$\begin{aligned} |\sigma(t)| > |\sigma_W(t)| \quad & \text{if and only if} \quad 1 - 2\left(\frac{W(t)}{L(t)}\right) < \frac{\sigma_L(t)}{\sigma(t)} < 1 \\ & \text{if and only if} \quad \frac{\sigma_L(t)}{\sigma_W(t)} < 1 \quad \text{or} \quad \frac{\sigma_L(t)}{\sigma_W(t)} > 1 + 2\frac{S(t)}{L(t)}. \end{aligned} \tag{13}$$

[7] Simply, $\mathrm{cov}_t(\mathrm{d}W(t), \mathrm{d}z(t)) = \mathrm{cov}_t(\mathrm{d}S(t), \mathrm{d}z(t)) + \mathrm{cov}_t(\mathrm{d}L(t), \mathrm{d}z(t))$.

These conditions are driven by the relative responsiveness of human capital and stock price to economic shocks, and by the relative importance of human capital in the total wealth of the economy. If $\sigma_L = \sigma$, all three processes respond to the shocks proportionately ($\sigma_W = \sigma = \sigma_L$), financial shocks being absorbed 'equally' by the total wealth and human capital. If $1 - 2W/L < \sigma_L/\sigma < 1$, the consumer is using his human capital as a cushion. For example, in the case of a bad shock, he does not allow his human capital to decrease proportionately, or he may even increase his human capital to compensate, and hence total wealth volatility is reduced. In the extreme case, when $\sigma_L/\sigma = 1 - W/L < 0$ and $\sigma_W = 0$, the consumer is absorbing all the financial shocks into his human capital. If $\sigma_L > \sigma$, the consumer is using his human capital to amplify the financial shocks. If $\sigma_L/\sigma < 1 - 2W/L < 0$, the consumer adjusts his human capital to cushion the shock, but in fact overcompensates so that his human capital volatility now dominates, making total wealth again more volatile than the stock price return.

Since L is the present value of future labor and W is the present value of future consumption, in general the comovements of labor and consumption, or σ_ℓ/σ_c, will be a factor in determining the relative volatilities of total wealth and stock price. This connection is made most clear by specializing to the case of logarithmic utility with labor and consumption driven by a geometric Brownian motion state variable, ε. In a log utility model with no labor ($u(c) = \log c$) the total wealth, consumption, and stock price volatilities are all equated, $\sigma = \sigma_W = \sigma_c$. In the presence of human capital, with $u(c,h) = \beta \log c + (1-\beta)\log h$, $\beta \in (0,1)$, total wealth and consumption volatilities remain equated, while human capital drives a wedge between the consumption and stock market volatilities:

$$\sigma(t) = \sigma_c(t) - \frac{L(t)}{W(t) - L(t)}(\sigma_L(t) - \sigma_c(t)), \tag{14}$$

where $W(t) = \beta(T-t)\,c(t)$ and

$$L(t) = (1-\beta)\,c(t)\,\mathrm{E}\left[\int_t^T \frac{\ell(s)}{\bar{\ell} - \ell(s)}\,\mathrm{d}s \,|\, \mathscr{F}_t\right]. \tag{15}$$

Proposition 1 states sufficient conditions for $|\sigma|$ or $|\sigma_c|$ to be lower than the other, based only on how consumption and labor move together in equilibrium.

Proposition 1. *Consider an economy consisting of one representative consumer–worker with utility* $u(c,h) = \beta \log c + (1-\beta)\log h$, $\beta \in (0,1)$. *Assume that the consumer's consumption and labor at time t are driven solely by a state variable* $\varepsilon(t)$, *i.e.*, $c(t) = c(\varepsilon(t), t)$, $\ell(t) = \ell(\varepsilon(t), t)$. *Further assume* ε *follows a* *geometric Brownian motion process. Then*

(a) $|\sigma(t)| > |\sigma_c(t)|$ *if* $\dfrac{\sigma_\ell(s)}{\sigma_c(t)} < 0$, $s \in [0, T]$,

(b) $|\sigma(t)| < |\sigma_c(t)|$ *if* $0 < \dfrac{\sigma_\ell(s)}{\sigma_c(t)} < \dfrac{2(\bar{\ell} - \ell(s))(\beta\bar{\ell} - \ell(s))}{(1-\beta)\bar{\ell}\,\ell(s)}$, $s \in [0,T]$,

(c) $|\sigma(t)| > |\sigma_c(t)|$ *if* $\dfrac{\sigma_\ell(s)}{\sigma_c(t)} > \dfrac{2(\bar{\ell} - \ell(s))(\beta\bar{\ell} - \ell(s))}{(1-\beta)\ \bar{\ell}\,\ell(s)}$, $s \in [0,T]$.

Cases (a) and (c) correspond to a 'cushioning' of the shock by the human capital, while case (b) corresponds to an 'amplification' or an 'overcushioning'. Case (a), while consistent with the financial data, is inconsistent with the macroeconomic data for how labor and consumption move together. Case (b) is inconsistent with the financial data. Case (c) is the only one reconcilable with both aspects of the data.

2.4. Equity risk premium

From a partial equilibrium perspective, but anticipating equilibrium, we may also evaluate the impact of the presence of labor on the consumption-based CAPM (CCAPM) (Breeden, 1979). Applying Itô's lemma to the consumer's first-order condition (7), using the fact that the state price density follows (2), and matching diffusion terms, implies an exact 2-factor version of the CCAPM that an equilibrium must satisfy:

$$\mu(t) - r(t) = -\frac{u_{cc}(t)}{u_c(t)}\operatorname{cov}\left(\frac{dS(t)}{S(t)}, d\hat{c}(t)\right) + \frac{u_{ch}(t)}{u_c(t)}\operatorname{cov}\left(\frac{dS(t)}{S(t)}, d\hat{\ell}(t)\right), \tag{16}$$

where $u_c(t)$ and its derivatives are shorthand for $u_c(\hat{c}(t), \bar{\ell} - \hat{\ell}(t))$ and its derivatives. The equity market risk premium depends on the covariance of its return with changes in aggregate employment as well as with changes in aggregate consumption. We are interested in conditions under which labor acts in the direction of resolution of the equity risk premium puzzle. In other words, when does the additional term in Eq. (16) lead to a higher risk premium prediction for given stock volatility and risk aversion. The comovement of labor and consumption is again a driving factor, as can be seen by rearranging Eq. (16) as

$$\mu(t) - r(t) = -\frac{u_{cc}(t)}{u_c(t)}\sigma(t)\,\sigma_c(t)\left[1 - \frac{u_{ch}(t)}{u_{cc}(t)}\frac{\sigma_\ell(t)}{\sigma_c(t)}\right]. \tag{17}$$

If u_{ch} is positive (negative) and labor and consumption move in the same (opposite) direction, Eq. (17) implies that the benchmark model with no labor is biased towards underestimating the risk premium of an asset covarying positively with aggregate consumption. On the other hand, if u_{ch} is positive (negative) and labor and consumption move oppositely (together), the benchmark model overestimates the risk premium as compared with our labor model. In the presence of labor, an asset covarying positively with the aggregate consumption may even yield a negative risk premium. Again, Sections 4–6 allow us to investigate the equilibrium comovement of labor and consumption.

3. The economic setting for a firm with labor demand

The representative firm in this economy faces the same information structure and set of securities as the consumer. At each time t, the firm uses labor, $\ell^D(t)$, as its only input to a production technology f which provides consumption good as output. The technology is stochastic, driven by a shock process ε, assumed (without loss of generality) to be positive and to follow an Itô process:

$$d\varepsilon(t) = \varepsilon(t)[\mu_\varepsilon(t)\,dt + \sigma_\varepsilon(t)\,dz(t)], \tag{18}$$

where the mean growth μ_ε and volatility σ_ε of the shock process are (possibly path-dependent) adapted processes. In some subsequent analysis (parts of Sections 5 and 6), we specialize the shock to be geometric Brownian motion, i.e., μ_ε, σ_ε are constants. The firm's output at time t is given by $f(\ell^D(t), \varepsilon(t))$. We assume f is increasing and concave in its first argument and that $\lim_{\ell^D \to 0} f_\ell(\ell^D, \varepsilon) = \infty$ and $\lim_{\ell^D \to 0} f(\ell^D, \varepsilon) \geq 0$.

Models of production economies, especially in the business cycle literature, typically assume the production shock to appear multiplicatively, i.e., $f(\ell^D, \varepsilon) = \varepsilon g(\ell^D)$. When the shock contains a long-term growth component, this assumption avoids long-term growth in labor demand over time, so as to be consistent with a stable economy. In our model of (short-term) financial fluctuations only, we allow the shock to appear more generally. There are two reasons why it is too restrictive to assume that shocks appear only multiplicatively. First, as will become clear, a multiplicative shock combined with Cobb–Douglas utility yields nonstochastic labor in equilibrium, which is too limiting to investigate the effects of labor. Second, a multiplicative shock exhibits $f_\varepsilon > 0$ and $f_{\ell\varepsilon} > 0$, capturing only technological improvements which also improve labor efficiency, while excluding the other realistic case of technological improvements which reduce labor productivity ($f_{\ell\varepsilon} < 0$).

We sometimes assume that f is of the CES form

$$f(\ell, \varepsilon) = \frac{1}{\alpha}[\eta\,\ell^\nu + (1-\eta)\varepsilon^\nu]^{\alpha/\nu}, \quad \alpha \in (0,1),\ \nu < 1,\ \eta \in (0,1).$$

This production function may be familiar from the context of a production technology with two inputs. However, our use of this form is quite nonstandard in that one of the arguments is ε, the exogenous shock, as opposed to an endogenous input, say, capital. Interpreting ε as an exogenous 'input' allows the interpretation of the parameters η and v. The parameter η represents the labor 'share' used by the firm as input. The parameter v is a measure of the elasticity of substitution of labor for shock, the measure of how easily a firm can 'substitute' into labor if a bad shock occurs. The CES production function has the property that the elasticity of the marginal product with respect to output in response to a shock is a constant,

$$\left.\frac{\partial(\log f_\ell)}{\partial(\log f)}\right|_\ell = \frac{f_{\ell\varepsilon} f}{f_\ell f_\varepsilon} = 1 - \frac{v}{\alpha}. \tag{19}$$

However, unlike the multiplicative shock case ($f_{\ell\varepsilon}f/f_\ell f_\varepsilon = 1$), this constant can take on all values in $(-\infty, \infty)$. The function has $f_\varepsilon > 0$ in all cases, and $f_{\ell\varepsilon} > 0$ if and only if $\alpha > v$. Hence, this family of production technologies can incorporate cases where production and productivity are affected in the same or in opposite directions by a shock.

The CES family includes a multiplicative shock as a special case, when $v = 0$:

$$f(\ell, \varepsilon) = \frac{1}{\alpha}[\ell^\eta \varepsilon^{1-\eta}]^\alpha.$$

Moreover, if $v/\alpha = 1$, we have

$$f(\ell, \varepsilon) = \frac{\eta}{v}\ell^v + \frac{(1-\eta)}{v}\varepsilon^v,$$

which is a case of an 'additive shock', where the shock can be interpreted as providing an additional endowment as in a Lucas (1978)-type pure-exchange economy.

The firm pays out a wage $w(t)$ for each unit of labor it utilizes, so its time-t profit is $f(\ell^D(t), \varepsilon(t)) - w(t)\ell^D(t)$, all of which it pays out as dividends $\delta(t)$ to its shareholders. The firm's objective is to maximize its market value, or the present value of its expected profits:

$$\max_{\ell^D} \mathrm{E}\left[\int_0^T \xi(t)\,(f(\ell^D(t), \varepsilon(t)) - w(t)\,\ell^D(t))\,\mathrm{d}t\right]. \tag{20}$$

The first-order condition of (20) is

$$f_\ell(\hat{\ell}^D(t), \varepsilon(t)) = w(t), \tag{21}$$

so that the optimal dividend (profit) process is given by

$$\hat{\delta}(t) = f(\hat{\ell}^D(t), \varepsilon(t)) - f_\ell(\hat{\ell}^D(t), \varepsilon(t))\,\hat{\ell}^D(t). \tag{22}$$

The assumption $\lim_{\ell^D \to 0} f_\ell(\ell^D, \varepsilon) = \infty$ ensures that $\hat{\ell}^D(t) \geq 0$, and the assumptions $\lim_{\ell^D \to 0} f(\ell^D, \varepsilon) \geq 0$ and $f_{\ell\ell} < 0$ ensure that $\hat{\delta}(t) > 0$.

4. Equilibrium in a consumer–worker economy

We assume there is one representative consumer–worker and one representative firm in the economy, and define equilibrium through clearing in the consumption good and labor markets. We note that the primitive uncertainty in this economy enters through the production technology of the firm, driven by the 'shock' process ε.

Definition. An equilibrium in an economy of one representative firm and one representative consumer–worker is a set of price processes, (r, S, w) and choice processes $(c^*, \ell^*, \ell^{D*}, \pi^*, \delta^*)$ such that (i) the consumer chooses his optimal consumption-labor-portfolio policy at the given price and wage processes, (ii) the firm chooses its optimal labor demand and dividend at the given wage process, and (iii) the consumption, labor, and security markets are cleared:

$$c^*(t) = \delta^*(t) + w(t)\ell^*(t) = f(\ell^{D*}(t),\ \varepsilon(t)),$$
$$\ell^{D*}(t) = \ell^*(t), \quad \pi^*(t) = S(t), \quad F^*(t) = S(t). \tag{23}$$

Proposition 2 presents conditions for equilibrium, allowing subsequent analysis of all endogenous fluctuations of interest.[8]

Proposition 2. If an equilibrium exists, the equilibrium labor $\ell^* = \ell^{D*}$ is given by

$$f_\ell(\ell^*(t), \varepsilon(t)) = \frac{u_h(f(\ell^*(t), \varepsilon(t)), \bar{\ell} - \ell^*(t))}{u_c(f(\ell^*(t), \varepsilon(t)), \bar{\ell} - \ell^*(t))} \tag{24}$$

and the equilibrium consumption and dividend processes by

$$c^*(t) = f(\ell^*(t), \varepsilon(t)), \tag{25}$$

$$\delta^*(t) = f(\ell^*(t), \varepsilon(t)) - f_\ell(\ell^*(t), \varepsilon(t))\ell^*(t). \tag{26}$$

The equilibrium state price density and wage processes are given by

$$\xi(t) = u_c(f(\ell^*(t), \varepsilon(t)), \bar{\ell} - \ell^*(t)), \tag{27}$$

$$w(t) = f_\ell(\ell^*(t), \varepsilon(t)), \tag{28}$$

[8] Establishing existence of equilibrium would involve: (i) imposing appropriate regularity conditions for all processes (including endogenous ones), expectations and optimization problems to be well-defined, (ii) showing there exists an optimal $(c,\ \ell,\ \ell^D,\ \pi,\ \delta)$ satisfying the assumed regularity conditions and clearing all markets, and (iii) establishing that all associated price processes do indeed satisfy the posited regularity conditions. Given our focus is on characterization, with particular attention to pertinent volatilities, we avoid step (i), and also step (iii) which would require derivation of price parameters of less immediate interest, such as r and μ. However, we note that the main additional element in showing existence over a benchmark model is to solve for the equilibrium labor from Eq. (24), which is indeed shown to have a unique interior solution in the Appendix.

and the equilibrium interest rate and stock price by

$$r(t) = -\frac{\mathscr{D}(u_c(t))}{u_c(t)}, \tag{29}$$

$$S(t) = \frac{1}{u_c(t)} \mathrm{E}\left[\int_t^T u_c(s)\{ f(\ell^*(s), \varepsilon(s)) - f_\ell(\ell^*(s), \varepsilon(s))\ell^*(s)\} \, \mathrm{d}s \,|\, \mathscr{F}_t \right], \tag{30}$$

where $\mathscr{D}(\cdot)$ *denotes the drift of the process in its argument, and* $u_c(t)$ *is shorthand for* $u_c(f(\ell^*(t), \varepsilon(t)), \bar{\ell} - \ell^*(t))$. *Consequently, S inherits an Itô process representation* (2), *as do* ℓ, c, δ, ξ *and w.*

Proposition 2 presents a fully analytical characterization of the equilibrium, with Eq. (24) determining the labor as a function of the shock ε and then Eqs. (25)–(30) determining all other quantities. Eq. (24) is shown in the Appendix to have a unique interior solution, which (along with its volatility σ_ℓ) may be highly nonlinear (possibly path-dependent) in the underlying Brownian uncertainty. This implicit determination of equilibrium labor adds an extra layer to the solution over that of a standard pure-exchange model solved analogously (employing martingale techniques); in the latter case, consumption and the state price density are expressed explicitly in terms of the exogenous dividend. In economic terms, Eq. (24) states that the marginal rate of substitution between consumption and leisure is equated to the marginal product of labor.

In a standard model with no labor, the equilibrium condition for clearing in the consumption good market is $c = \delta$; the consumption absorbs all the financial shocks generated by the exogenously given dividends ($\sigma_c = \sigma_\delta$). In our labor model we have $c = \delta + w\ell$; the financial dividend shocks are now absorbed into both consumption and labor income, driving a wedge between consumption and dividend volatilities. Proposition 3 summarizes the responsiveness to a shock of pertinent processes chosen by the consumer or firm, as well as of the state price density process, by reporting the betas between these processes and the shock (e.g., $\beta_{\ell,\varepsilon} = \mathrm{cov}(\mathrm{d}\ell^*, \mathrm{d}\varepsilon)/\mathrm{var}(\mathrm{d}\varepsilon) = \sigma_\ell/\sigma_\varepsilon$).[9]

Proposition 3. *If an equilibrium exists, the volatility of labor is given by*

$$\frac{\sigma_\ell(t)}{\sigma_\varepsilon(t)} = V(t)\frac{\varepsilon(t) f_\varepsilon(t)}{\ell(t) f(t)}\left\{ \frac{f_{\ell\varepsilon}(t) f(t)}{f_\ell(t) f_\varepsilon(t)} - \frac{1}{e_{c,h}(t)} \right\}. \tag{31}$$

[9] We normalize all volatilities by the shock volatility to abstract away from the arbitrariness in the sign of σ_ε (due to the arbitrariness in the sign of dz).

Consequently, the volatilities of consumption, wage, dividend, and state price density satisfy:

$$\frac{\sigma_c(t)}{\sigma_\varepsilon(t)} = \varepsilon(t)\frac{f_\varepsilon(t)}{f(t)} + V(t)\varepsilon(t)\frac{f_\varepsilon(t)f_\ell(t)}{f(t)^2}\left\{\frac{f_{\ell\varepsilon}(t)f(t)}{f_\ell(t)f_\varepsilon(t)} - \frac{1}{e_{c,h}(t)}\right\}, \tag{32}$$

$$\frac{\sigma_w(t)}{\sigma_\varepsilon(t)} = \varepsilon(t)\frac{f_{\ell\varepsilon}(t)}{f_\ell(t)} + V(t)\,\varepsilon(t)\frac{f_\varepsilon(t)f_{\ell\ell}(t)}{f(t)f_\ell(t)}\left\{\frac{f_{\ell\varepsilon}(t)f(t)}{f_\ell(t)f_\varepsilon(t)} - \frac{1}{e_{c,h}(t)}\right\}, \tag{33}$$

$$\frac{\sigma_\delta(t)}{\sigma_\varepsilon(t)} = \varepsilon(t)\frac{f_\varepsilon(t) - \ell(t)f_{\ell\varepsilon}(t)}{f(t) - \ell(t)f_\ell(t)}$$

$$- V(t)\,\ell(t)\,\varepsilon(t)\frac{f_\varepsilon(t)f_{\ell\ell}(t)}{f(t)(f(t) - \ell(t)f_\ell(t))}\left\{\frac{f_{\ell\varepsilon}(t)f(t)}{f_\ell(t)f_\varepsilon(t)} - \frac{1}{e_{c,h}(t)}\right\}, \tag{34}$$

$$\frac{\theta(t)}{\sigma_\varepsilon(t)} = -\frac{u_{cc}(t)}{u_c(t)}\varepsilon(t)f_\varepsilon + V(t)\,\varepsilon(t)\frac{f_\varepsilon(t)}{f(t)^2}\frac{1}{e_{c,h}(t)}\left\{\frac{f_{\ell\varepsilon}(t)f(t)}{f_\ell(t)f_\varepsilon(t)} - \frac{1}{e_{c,h}(t)}\right\}, \tag{35}$$

where

$$V(t) \equiv -\frac{u_c(t)^2\,u_h(t)}{\{u_h(t)^2\,u_{cc}(t) + u_c(t)^2\,u_{hh}(t) - 2u_c(t)\,u_h(t)\,u_{ch}(t) + u_c(t)^3 f_{\ell\ell}(t)\}} > 0,$$

$$e_{c,h}(t) \equiv \frac{u_h(t)\,u_c(t)}{c(t)[u_{ch}(t)\,u_c(t) - u_h(t)\,u_{cc}(t)]} > 0,$$

the consumer's elasticity of substitution of leisure for consumption, and where $u(t)$, $f(t)$, *and their derivatives are shorthand for* $u(c^*(t), \bar{\ell} - \ell^*(t))$, $f(\ell^*(t), \varepsilon(t))$, *and their derivatives.*

The expressions $\{f_{\ell\varepsilon}f/f_\ell f_\varepsilon - 1/e_{c,h}\}$ and f_ε are important in determining how labor responds to a shock. There are two potentially offsetting effects on a consumer's labor when a shock occurs: the wage increases (or decreases) causing the agent to work more (or less); but the output increases (or decreases) tending to make the consumer substitute into (or out of) leisure. The factor $f_{\ell\varepsilon}f/f_\ell f_\varepsilon$ quantifies whether the wage or the output is impacted more, as it is the ratio of the percentage change in productivity to the percentage change in production in response to a shock (keeping labor fixed). The factor $1/e_{c,h}$ controls the degree of substitutability between leisure and consumption, as it is the percentage change in the consumer's marginal rate of substitution between consumption and leisure per unit change in the consumption-to-leisure ratio, keeping leisure fixed. Hence, if $f_{\ell\varepsilon}f/f_\ell f_\varepsilon = 1/e_{c,h}$, when a shock occurs, if the consumer keeps his labor unchanged, he is still at the optimum at the new output and wage; his marginal rate of substitution and the wage change by the same percentage so that they are still equated to each other.[10] The shocks to wages and output counterbalance each other. However, if $f_{\ell\varepsilon}f/f_\ell f_\varepsilon > 1/e_{c,h}$ and $f_\varepsilon > 0$, in response to a shock, the wage increases by more than the marginal rate of substitution, keeping labor fixed. Hence, the change in productivity dominates and the consumer will choose to work harder (substitute out of leisure into consumption), giving $\sigma_\ell/\sigma_\varepsilon > 0$. Other cases can be explained analogously. Eq. (31) can alternatively be explained from the firm's point of view, again by observing that a shock has two effects on labor demand: the output is increased (or decreased), making workers better (or worse) off so that the firm has to pay more (or less) for labor and hence its labor demand decreases (or increases); and simultaneously productivity increases (or decreases) so the firm's labor demand increases (or decreases).

The effect of a shock on consumption, wages, dividends, and state prices acts in two ways, through a 'direct' effect, and an 'indirect' effect due to the labor response. In the case of consumption, the direct effect of a shock is the increase in output for a fixed quantity of labor (represented by a nonzero f_ε). The indirect effect occurs because if a consumer increases (decreases) his labor, he substitutes into (out of) consumption. These two effects are represented by the two terms in Eq. (32). In the case of wages, the direct effect is due to the change in productivity, for a fixed quantity of labor (represented by a nonzero $f_{\ell\varepsilon}$); the indirect effect occurs because if the consumer increases (decreases) his labor, this will tend to push the wage up (down).

According to the discussion of Sections 2.3 and 2.4, we are particularly interested in the sign of σ_ℓ/σ_c, i.e., whether consumption and labor are positively or negatively related. The following corollary is deduced from Proposition 3.

Corollary 1.

$$\frac{\sigma_\ell(t)}{\sigma_c(t)} > 0 \quad \textit{if}\ \frac{f_{\ell\varepsilon}(t)f(t)}{f_\ell(t)f_\varepsilon(t)} > \frac{1}{e_{c,h}(t)};$$

$$\frac{\sigma_\ell(t)}{\sigma_c(t)} < 0 \quad \textit{if}\ \frac{1}{e_{c,h}(t)} > \frac{f_{\ell\varepsilon}(t)f(t)}{f_\ell(t)f_\varepsilon(t)} \geq 0.$$

Corollary 1 presents a sufficient condition for σ_ℓ/σ_c to be positive, as observed in the data. This occurs when the shock directly affects both production and productivity in the same direction, but the direct effect on wages is higher than on the marginal rate of substitution, so as to induce labor also to move in the

[10] This intuition is somewhat related to a well-known (partial equilibrium) result of the Slutsky equation, presented, for example, by Nicholson (1992), (pp. 687–689), that labor does not change in response to a wage change when there is no nonlabor income and the utility is Cobb–Douglas ($e_{c,h} = 1$). However, what we describe here is different since the shock affects both the wage and output simultaneously. We are also looking at a general equilibrium rather than a partial equilibrium result.

same direction. Then consumers work less rather than more at times of low consumption. Corollary 1 also presents a sufficient condition for σ_ℓ/σ_c to be negative, as in Bodie et al.'s conjecture. If $1/e_{c,h} > 0 > f_{\ell\varepsilon}f/f_\ell f_\varepsilon$, then the sign of σ_ℓ/σ_c remains ambiguous. This is when the shock affects production and productivity in opposite directions.

5. Cases in which deterministic labor arises in equilibrium

Proposition 4 reports equivalent conditions for employment to be locally deterministic and all unambiguous implications that may be deduced in such cases.

Proposition 4. *Assume* $f_\varepsilon(t) \neq 0$. *For a given time and state*:

$$\text{(a) } \sigma_\ell(t) = 0 \quad \textit{if and only if} \quad \text{(b) } \frac{f_{\ell\varepsilon}(t)f(t)}{f_\varepsilon(t)f_\ell(t)} = \frac{1}{e_{c,h}(t)}.$$

Conditions (a) *or* (b) *also imply*

$$\text{(c) } \sigma_c(t) = \frac{f_\varepsilon(t)}{f(t)}\sigma_\varepsilon(t)\,\varepsilon(t), \quad \text{(d) } \sigma_w(t) = \frac{f_{\ell\varepsilon}(t)}{f(t)}\sigma_\varepsilon(t)\,\varepsilon(t),$$

$$\text{(e) } \sigma_\delta(t) = \frac{f_\varepsilon(t) - \ell(t)f_{\ell\varepsilon}(t)}{f(t) - \ell(t)f_\ell(t)}\sigma_\varepsilon(t)\,\varepsilon(t), \textit{ and}$$

$$\text{(f) } \theta(t) = -\frac{u_{cc}(t)}{u_c(t)}c(t)\,\sigma_c(t) = -\frac{u_{cc}(t)}{u_c(t)}c(t)\frac{f_\varepsilon(t)}{f(t)}\sigma_\varepsilon(t)\,\varepsilon(t).$$

In particular,

$$\textit{if} \quad \frac{f_{\ell\varepsilon}(t)f(t)}{f_\varepsilon(t)f_\ell(t)} = \frac{1}{e_{c,h}(t)} = 1 \quad \textit{then } \sigma_w(t) = \sigma_c(t) = \sigma_\delta(t);$$

$$\textit{if} \quad \frac{f_{\ell\varepsilon}(t)\;f(t)}{f_\varepsilon(t)f_\ell(t)} = \frac{1}{e_{c,h}(t)} < 1 \quad \textit{then } |\sigma_\delta(t)| > |\sigma_c(t)| > |\sigma_w(t)|;$$

$$\textit{if} \quad \frac{f_{\ell\varepsilon}(t)f(t)}{f_\varepsilon(t)\;f_\ell(t)} = \frac{1}{e_{c,h}(t)} > 1$$

$$\textit{then} \begin{cases} |\sigma_w(t)| > |\sigma_c(t)| > |\sigma_\delta(t)| & \textit{for } \dfrac{f(t)}{\ell(t)f_\ell(t)} > \dfrac{1 + e_{c,h}(t)}{2e_{c,h}(t)}, \\[2ex] |\sigma_w(t)| > |\sigma_\delta(t)| > |\sigma_c(t)| & \textit{for } \dfrac{1 + e_{c,h}(t)}{2e_{c,h}(t)} > \dfrac{f(t)}{\ell(t)f_\ell(t)} > \dfrac{2}{1 + e_{c,h}(t)}, \\[2ex] |\sigma_\delta(t)| > |\sigma_w(t)| > |\sigma_c(t)| & \textit{for } \dfrac{2}{1 + e_{c,h}(t)} > \dfrac{f(t)}{\ell(t)f_\ell(t)}. \end{cases}$$

When labor is locally deterministic, there remains only the 'direct' effect of the shock on consumption, wages, dividends, and state prices, thus facilitating a comparison of the predicted volatilities. The factor $f_{\ell\varepsilon} f / f_\ell f_\varepsilon$ determines whether the shock is absorbed more into wages or into consumption. If $f_\varepsilon / f > f_{\ell\varepsilon} / f_\ell$, the shock affects output more than it does marginal productivity, yielding more volatility in consumption than in wages, and vice versa if $f_{\ell\varepsilon} / f_\ell > f_\varepsilon / f$. Even when deterministic, the presence of labor drives a wedge between consumption and dividend volatilities; if wage income cushions the dividend shocks, consumption volatility is reduced relative to the dividend volatility, and vice versa if wage income amplifies the shocks. The dividend volatility may lie on either side of or between consumption and wage volatilities. Proposition 4 provides one set of conditions under which the dividend, wage, and consumption volatilities are ranked as observed empirically ($|\sigma_\delta| > |\sigma_c| > |\sigma_w|$).

The case of deterministic labor is of no use in explaining the equity risk premium puzzle, since the equity risk premium expression (16) collapses to a standard one-factor consumption CAPM. For given stock volatility and risk aversion, the model then predicts a risk premium identical to the benchmark model. However, even when labor is deterministic, the human capital volatility σ_L is still nonzero, implying that financial shocks are not fully absorbed by total wealth ($\sigma \neq \sigma_W$). In fact, the case of nonstochastic labor provides a case (beyond log utility) where we may state sufficient conditions for the stock price volatility to be higher than total wealth volatility, as reported in Proposition 5.

Proposition 5. *Assume* $\sigma_\ell(t) = 0$, $t \in [0, T]$, *and* ε *follows a geometric Brownian motion. Then*

$$|\sigma(t)| > |\sigma_W(t)| \quad \textit{if} \; \frac{\sigma_c(s)}{\sigma_c(t)} > 0, \quad -\frac{u_{cc}(t)}{u_c(t)} c(t) < 1, \quad u_{ch}(t) < 0, s, t \in [0, T].$$

Condition (b) in Proposition 4 is met if the utility function and the production technology are both of the CES form with $v/\alpha = \rho$. Indeed, for the most common case considered in the economics literature, especially in business-cycle models, Cobb–Douglas utility combined with multiplicative shocks, $f(\ell, \varepsilon) = \varepsilon\, g(\ell)$, we obtain exactly

$$\frac{f_{\ell\varepsilon}(t) f(t)}{f_\varepsilon(t) f_\ell(t)} = \frac{1}{e_{c,h}(t)} = 1, \quad t \in [0, T]. \tag{36}$$

This case can be considered as a kind of 'benchmark', since labor is deterministic and the model implications coincide with those of a model with no labor, as reported in Proposition 6.

Proposition 6. *Assume* $u(c, h) = (1/\gamma)[c^\beta h^{1-\beta}]^\gamma$, $\gamma < 1$, $\beta \in (0, 1)$, *and* $f(\ell, \varepsilon) = \varepsilon\, g(\ell)$. *Assume* ε *follows a geometric Brownian motion process. Then*

$$\sigma_\ell(t) = 0,$$

$$\sigma(t) = \sigma_W(t) = \sigma_L(t) = \sigma_w(t) = \sigma_c(t) = \sigma_\delta(t) = \sigma_\varepsilon,$$

$$\theta(t) = (1 - \gamma\beta)\sigma_\varepsilon, \quad t \in [0, T].$$

In this benchmark case, shocks are absorbed equally by wage income and consumption and hence equally by human capital and total wealth. Accordingly, human capital, total wealth, and stock price all have identical volatility. Hence, for the most commonly considered case, the predictions, of deterministic labor and no discrepancy between consumption and dividend volatilities nor consumption and stock market volatilities, are entirely contrary to the data. Our model clearly pinpoints the need to deviate from this benchmark case in order to capture the effects of stochastic labor and to attempt to explain regularities in the data.

6. Deviations from the deterministic labor case

Proposition 3 and Section 5 reveal that if we assume both the production and the utility functions to be of the CES form but with $\alpha/v \neq \rho$, we will maintain tractability but allow for stochastic labor. We consider two particular deviations. First, we assume the simplest case of CES utility, log utility, and allow the production technology to have a general CES form. Second, we assume multiplicative shocks, but allow the utility function to have a general CES form. We report only (and all) results which have unambiguous directions, not only those consistent with data.

6.1. Log utility function and 'CES' production technology

Proposition 7 outlines how the various processes respond to a technological shock under the assumptions of this section. According to Proposition 4, we obtain stochastic labor unless $v/\alpha = 0$.

Proposition 7. *Assume* $f(\ell,\varepsilon) = (1/\alpha)[\eta\ell^{v} + (1-\eta)\varepsilon^{v}]^{\alpha/v}$, $\alpha \in (0,1)$, $v < 1$, $\eta \in (0,1)$ *and* $u(c,h) = \beta \log c + (1-\beta)\log h$, $\beta \in (0,1)$. *Assume* ε *follows a geometric Brownian motion.*[11] *Then for* $s,t \in [0,T]$, *we have*

	(a) $v > \alpha > 0$	(b) $\alpha \geq v > 0$	(c) $v < 0$
$\sigma_\ell(t)/\sigma_\varepsilon(t)$	$-$ ve	$-$ ve	$+$ ve
$\sigma_c(t)/\sigma_\varepsilon(t)$	$+$ ve	$+$ ve	$+$ ve
$\sigma_\ell(s)/\sigma_c(t)$	$-$ ve	$-$ ve	$+$ ve
$\sigma_w(t)/\sigma_\varepsilon(t)$	$> -(v-\alpha)$	$+$ ve	$\in (0, \alpha - v)$
$\theta(t)/\sigma_\varepsilon(t)$	$+$ ve	$+$ ve	$+$ ve
	—	$\lvert\sigma_c(t)\rvert > \lvert\sigma_w(t)\rvert$	$\lvert\sigma_c(t)\rvert > \lvert\sigma_w(t)\rvert$
	$\lvert\sigma(t)\rvert > \lvert\sigma_c(t)\rvert$	$\lvert\sigma(t)\rvert > \lvert\sigma_c(t)\rvert$	—

[11] All the results, except for $\sigma_\ell(s)/\sigma_c(t)$, $s \neq t$, and the $|\sigma|$, $|\sigma_c|$ comparisons, hold more generally when ε is not a geometric Brownian motion.

Proposition 7 reveals unambiguous conclusions about the direction of the response of consumption and labor to shocks, and about the comovement of consumption and labor. In case (a), since $f_\varepsilon > 0$ and $f_{\ell\varepsilon} < 0$, the direct effect of a shock is to increase output, hence increasing the marginal rate of substitution between consumption and leisure, but to decrease wages. As a result the wage will be 'too low' from the consumer's point of view and he will work less, leading to $\sigma_\ell/\sigma_\varepsilon < 0$. In cases (b) and (c), the direct effect of a shock is to increase both the output (and hence marginal rate of substitution) and the wage. If $f_{\ell\varepsilon}f/f_\ell f_\varepsilon < 1/e_{c,h}$ (case (b)), the increase in the marginal rate of substitution dominates, leading the consumer to work less. In case (c) the wage increase dominates, leading the consumer to work more. We are, then, able to derive cases which both agree with and run counter to the Blanchard and Fisher (1989) argument of a negative comovement of labor and consumption. Explained in terms of the firm, when the firm has a high elasticity of substitution of labor for shock ($v > 0$) it will substitute out of labor when a good shock occurs, yielding $\sigma_\ell/\sigma_\varepsilon < 0$. The reverse is true when the firm has a low substitution of labor for shock ($v < 0$). Proposition 7 reveals that, independently of the parameters v and α, the direct effect of a 'good' shock on output is always an increase; no matter what the indirect effect of the varying labor is on consumption, the net effect of a shock is always to increase consumption, since $\sigma_c/\sigma_\varepsilon > 0$.

In the case of log utility, according to Proposition 1 we can use the behavior of labor and consumption to address the excess volatility of stock price returns. For the regions $v > 0$, contrary to the data, comovements of labor and consumption are always negative. In Section 2.3 we outlined how this behavior implies that the human capital is being used to 'cushion' the financial shocks in the economy. Hence consumption volatility is indeed predicted to be lower than the stock price volatility. For the other region $v < 0$, comovements of labor and consumption are positive, consistent with the macroeconomic data. This behavior is consistent with consumption being either more or less volatile than the stock price. One way (not yet achieved) to derive the observed case of $|\sigma| > |\sigma_c|$ would be to bound σ_ℓ/σ_c from below. The ambiguity in case (c) suggests that human capital may instead sometimes be used to amplify the shocks, yielding a stock price less volatile than consumption. We note that all unambiguous results in Proposition 7 are consistent with the data, except for the negative comovement of consumption and labor in cases (a) and (b).

We mentioned in Section 3 that $v/\alpha = 1$ in the CES production technology yields one type of so-called 'additive shocks' (i.e., $f(\ell, \varepsilon) = g_1(\ell) + g_2(\varepsilon)$).[12] The shock then acts as an additional (exogenous) endowment in consumption good

[12] Although we derive Proposition 8 as a special case of the CES production function, most of the results therein extend to more general cases of additive shocks $f(\ell, \varepsilon) = g_1(\ell) + g_2(\varepsilon)$, where $g_1' > 0$, $g_1'' < 0$, $g_2' > 0$. Results are unchanged except for the computation of σ_ℓ and σ_w magnitudes, which becomes $|\sigma_\ell| > |\sigma_w|$ if and only if $-g_1''\ell/g_1' < 1$.

to the firm, affecting production output, but not directly affecting productivity. This production technology has $f_\varepsilon = (1-\eta)\varepsilon^{\nu-1} > 0$, $f_{\ell\varepsilon} = 0$, and $f_{\ell\varepsilon} f / f_\ell f_\varepsilon = 0 < 1/e_{c,h}$. Proposition 8 reports this case whose results admit a simpler intuition.

Proposition 8. *Assume* $u(c,h) = \beta \log c + (1-\beta)\log h$, $\beta \in (0,1)$, *and*

$$f(\ell,\varepsilon) = \frac{\eta}{\nu}\ell^\nu + \frac{(1-\eta)}{\nu}\varepsilon^\nu,$$

$\nu < 1$, $\eta \in (0,1)$. *Assume* ε *follows a geometric Brownian motion. Then for* $s, t \in [0,T]$, *the properties of Proposition* 7 *case* (*b*) *are satisfied and additionally,*

$$|\sigma_\ell(t)| > |\sigma_w(t)|.$$

The intuition behind the labor, consumption, and wage responses to a shock is a special case of that given for Proposition 7(b). Thinking of the shock as an endowment, intuitively if the consumer has more exogenous endowment, he is going to consume more and not need to work as much, so the firm will have to offer higher wages. Labor and consumption always respond in opposite directions to a shock, so human capital is cushioning the financial shocks and causing stock price volatility to be higher than consumption volatility. By specializing to this additive shock, we are also able to predict that labor will be more volatile than wages (further strengthening consistency with the data).

All the results of this subsection, except the stock market and consumption volatility comparisons, go through for a log-power utility function of the form $u(c,h) = \beta \log c + (1-\beta)h^\gamma/\gamma, \gamma \leq 1$. The special case of $\gamma = 1$ has been used by several authors to capture the notion of indivisible labor, as discussed in Remark 1.

Remark 1 (*Indivisible labor*). Indivisible labor is the most commonly proposed notion to reconcile the apparently contradictory observations of highly volatile labor relative to wages, while individual agents exhibit preferences with relatively low intertemporal substitution of leisure (across time). Hansen (1985) and Rogerson (1988) show that once an agent has decided to work, restricting him to work only for a fixed number of hours per period allows the generation of a representative agent with high intertemporal substitution of leisure, while maintaining individual agents with low substitution.

Following Hansen (1985) and Rogerson (1988), our model may also capture indivisible labor as a special case. We take a step back from our representative agent and assume a continuum of ex ante identical agents who have logarithmic preferences and who at each time t choose a probability over a binary leisure choice (working full time versus not working).[13] Aggregation of these agents generates a representative agent having a log-linear utility function of the form $u(c,h) = \beta \log c + (1-\beta)\,h$.[14] A relaxation of the strict indivisibility of labor can be informally captured by a log-power utility, $u(c,h) = \beta \log c + (1-\beta)h^{\gamma}/\gamma$, $\gamma \leq 1$ and γ 'close' to 1.

The standard intuition (e.g., Blanchard and Fisher, 1989, Section 7.2) connecting high labor volatility with high intertemporal substitution of leisure can be paraphrased in our setting via the log-power agent's first order conditions. Eq. (8) yields

$$h(t) = \left(\frac{1-\beta}{yw(t)\xi(t)}\right)^{1/(1-\gamma)}, \tag{37}$$

which by Itô's Lemma implies

$$\sigma_{\ell}(t) = \frac{1}{1-\gamma}\frac{h(t)}{\ell(t)}(\sigma_w(t) - \theta(t)). \tag{38}$$

At a partial equilibrium level, Eq. (38) reveals that as the intertemporal substitution of leisure increases ($\gamma \to 1$), labor volatility rapidly increases relative to wage volatility (maintaining all prices fixed). However, when prices are allowed to adjust, θ may adjust to offset the wage volatility, in which case Eq. (38) does not require labor volatility to grow with intertemporal substitution. A general equilibrium model is required to address this question, to which we now turn.

We find (details available upon request) that preferences of the form $u(c,h) = \beta \log c + (1-\beta)h^{\gamma}/\gamma$, $\gamma \leq 1$, yield properties similar to log–log preferences. Under multiplicative shocks, labor remains deterministic, thus clearly less volatile than wages. Under a general CES production function,

$$\frac{|\sigma_{\ell}(t)|}{|\sigma_w(t)|} = \frac{1}{\ell(t)}\left\{\frac{1}{\left(\frac{\alpha-v}{v}\right)\left[\frac{1-\beta}{\beta}\frac{1}{h(t)^{1-\gamma}} + \frac{1-\gamma}{h(t)}\right] - \frac{\alpha}{v}\frac{f_{\ell\ell}(t)}{f_{\ell}(t)}}\right\}, \tag{39}$$

[13] Each agent derives utility $\kappa \log c + (1-\kappa)\log h$, with h only taking on the two constant values $\bar{\ell} - \ell_1$ (working full time) or $\bar{\ell}$ (not working). To get around the nonconvexity implied by the binary choice of h, agents are assumed to choose the probability $p(t)$ of working, leading to an expected utility at time t of: $\kappa \log c(t) - p(t)(1-\beta)[\log(\bar{\ell} - \ell_1) - \log \bar{\ell}] + (1-\beta)\log \bar{\ell}$. $p(t)$ represents the fraction of the population working, and the aggregate hours worked is $\ell(t) = \bar{L} - h(t) = p(t)\ell_1$. This leads to a representative agent's utility function (ignoring constant terms) of $u(c(t),h(t)) = \kappa \log c(t) + (1-\kappa)\,\phi\, h(t), \phi \equiv [\log \bar{\ell} - \log(\bar{\ell} - \ell_1)]/\ell_1 > 0$. Indivisibility of labor may also be tractably incorporated with more general, separable, individual preferences, $v(c) + g(h)$, (Rogerson, 1988) yielding $u(c(t),h(t)) = v(c(t)) + \phi\, h(t), \phi \equiv [g(\bar{\ell}) - g(\bar{\ell} - \ell_1)]/\ell_1 > 0$, also linear in labor.

[14] This utility function has $\lim_{h \to 0} u_h(c,h) \neq \infty$, so the equilibrium leisure can become negative, but remains finite.

which does not unambiguously increase as $\gamma \to 1$, and appears to allow labor volatility to be either higher or lower than wage volatility. For example, under an additive shock,

$$|\sigma_\ell(t)| = \frac{1}{1-v}|\sigma_w(t)|, \qquad \gamma \le 1, \tag{40}$$

in which case relative labor volatility is independent of γ. Labor is more volatile than wages, but this effect is driven by the parameters of the production technology, not the agent's preferences. Furthermore, for a general CES production function, we show

$$\sigma_w(t) - \theta(t) = -\frac{v}{\alpha}\frac{\varepsilon(t) f_\varepsilon(t)}{f(t)}\left[\frac{(1-\gamma)/h(t)}{\dfrac{1-\beta}{\beta}\dfrac{1}{h(t)^{1-\gamma}} + \dfrac{1-\gamma}{h(t)} - \dfrac{f_{\ell\ell}(t)}{f_\ell(t)}}\right]\sigma_\varepsilon(t) \to 0$$

$$\text{as } \gamma \to 1. \tag{41}$$

Hence, Eq. (38) does not require arbitrarily large labor volatility.

The presence of θ in our model, then, breaks the connection between intertemporal leisure substitutability and high labor volatility. Hansen (1985) was able numerically to generate (and in fact overpredict) higher labor volatility than wage volatility. He attributed his overprediction to his assumption of extreme indivisibility of labor. A crucial difference between our model and Hansen's is that our agents face complete financial markets to trade their production output across states and times, leading to the presence of θ, the fluctuation in valuation of concurrent output. As an additional reason for Hansen's overprediction, we propose his limited financial markets for the firm's output. We conjecture that a model of indivisible labor combined with intermediate incomplete markets could well capture the excess labor volatility observed, but a detailed analysis of this point is beyond the scope of this paper.

6.2. CES utility function and multiplicative shocks

We assume $f(\ell,\varepsilon) = \varepsilon\, g(\ell)$, where $g(\cdot)$ is strictly increasing and concave in its argument and satisfies $g(0) > 0$, $\lim_{\ell\to 0} g'(\ell) = \infty$, implying $f_\varepsilon > 0, f_{\ell\varepsilon} > 0$ and $f_{\ell\varepsilon} f / f_\varepsilon f_\ell = 1$. According to Proposition 4, we obtain stochastic labor unless $\rho = 0$ (Cobb–Douglas). We deduce the following conclusions about the directions, bounds, and magnitudes of a response to a technological shock.

Proposition 9. *Assume* $f(\ell,\varepsilon) = \varepsilon\, g(\ell)$ *and* $u(c,h) = (1/\gamma)[\beta c^\rho + (1-\beta)h^\rho]^{\gamma/\rho}$, $\gamma < 1$, $\rho < 1$, $\beta \in (0,1)$. *Then we have the situation described in* Table 1

(i) *denotes condition* $1 - \ell(t)g'(t)/g(t) < -\ell(t)g''(t)/g'(t)$
(ii) *denotes condition* $1 - \ell(t)g'(t)/g(t) > -\ell(t)g''(t)/g'(t)$.

Table 1

	(a) $\rho > 0$		(b) $\rho < 0$	
	(i)	(ii)	(i)	(ii)
$\sigma_\ell(t)/\sigma_\varepsilon(t)$	$+$ ve		$-$ ve	
$\sigma_c(t)/\sigma_\varepsilon(t)$	$\in(1, 1/(1-\rho))$		$\in(1/(1-\rho), 1)$	
$\sigma_\ell(t)/\sigma_c(t)$	$+$ ve		$-$ ve	
$\sigma_w(t)/\sigma_\varepsilon(t)$	$\in(1-\rho, 1)$		$\in(1, 1-\rho)$	
$\sigma_\delta(t)/\sigma_\varepsilon(t)$	$\in(1, \infty)$		$-$ ve	
$\theta(t)/\sigma_\varepsilon(t)$	$\in(0, 1 - c(t)\, u_{cc}(t)/u_c(t))$		$\in(0, - c(t)\, u_{cc}(t)/u_c(t))$	
	$\lvert\sigma_\delta(t)\rvert > \lvert\sigma_c(t)\rvert$	$\lvert\sigma_\delta(t)\rvert < \lvert\sigma_c(t)\rvert$	$\lvert\sigma_\delta(t)\rvert < \lvert\sigma_c(t)\rvert$	$\lvert\sigma_\delta(t)\rvert > \lvert\sigma_c(t)\rvert$
	$\lvert\sigma_c(t)\rvert > \lvert\sigma_w(t)\rvert$		$\lvert\sigma_c(t)\rvert < \lvert\sigma_w(t)\rvert$	
	$\lvert\sigma_\delta(t)\rvert > \lvert\sigma_w(t)\rvert$		—	

For a multiplicative shock, Proposition 9 reveals that the comovement of labor and consumption depends on whether the consumer has an elasticity of substitution of leisure for consumption greater or less than 1. For high substitutability ($\rho > 0$), the comovement is positive (consistent with the data), while for low substitutability ($\rho < 0$) the comovement is negative. Under a multiplicative shock, the direct effect of a 'good' shock is to increase both output and productivity (or wages) proportionately. If $\rho > 0$ the marginal rate of substitution of leisure for consumption increases by less than output and wages, so consumers choose to work harder. If $\rho < 0$, the reverse is true. Proposition 9 also reveals that the total effect of a good shock is always to increase both consumption and wages. We may note that in case (a) (i) all unambiguous results are consistent with the macroeconomic data.

Proposition 9 compares the variability of wages, dividends, and consumption. Conditions are identified for consumption to be 'smoothed' relative to dividends, as seen in the data, or to be made more volatile relative to dividends. The volatility ranking of consumption and wages is determined solely by the utility function, with $\rho < 0$ yielding a higher wage volatility, and $\rho > 0$ yielding a lower wage volatility.

The unambiguous results on comovement of consumption and labor allow us to address the equity risk premium, as outlined in Section 2.4. For $\gamma > \rho > 0$, since $u_{ch} > 0$ and consumption and labor comove positively, Eqs. (16) and (17) imply that the benchmark model underestimates the equity risk premium relative to our model with labor. This is when the substitutability of leisure for consumption and intertemporal substitution are both high, but the latter dominates. When $\rho > 0$ but $\gamma < \rho$, since $u_{ch} < 0$ the equity risk premium decreases. Similarly, for $\gamma < \rho < 0$, the equity risk premium prediction is increased in our model, while for $\rho < 0$ but $\gamma > \rho$ the premium is decreased (for a given stock volatility and risk aversion). So when either $\gamma > \rho > 0$ or $\gamma < \rho < 0$, the inclusion of human capital acts in the direction of resolving the equity risk premium puzzle. In either case, the effects of a shock on leisure and on consumption reinforce each other in influencing the agent's marginal utility and hence impacting the pricing of risk.

7. Conclusion

We have developed a continuous-time general equilibrium model to adapt dynamic asset pricing theory to include human capital. In the presence of human capital, the stock market and total wealth volatilities are no longer equal, nor are the dividend and consumption volatilities. We have outlined general conditions on human capital volatility which determine whether its presence acts to smooth total wealth relative to the standard model, or acts to increase total wealth volatility. In the special case of log utility, these reduce to conditions on the comovements of aggregate consumption and labor supply. If labor and consumption always respond to a shock in opposite directions, consumption is unambiguously smoothed relative to the stock market as seen in the data. If labor and consumption respond in the same direction, consumption may be smoother or more volatile.

We provide analytical representations for all quantities and identify the factors determining the response of labor (and hence consumption, wages, and dividends) to a production shock: the representative consumer's elasticity of substitution of consumption for leisure, and the elasticity of the marginal product to output in response to a shock. For the example of log utility and 'constant elasticity of substitution' production technology, we apply this analysis to identify cases for which the effect of human capital is to unambiguously smooth aggregate consumption relative to the stock market.

The limitations of a multiplicative shock combined with Cobb–Douglas utility are made clear, since this combination yields deterministic labor. Deviations from these assumptions are analyzed. Overall, many of the tractable cases yield explicit results fully consistent with the data, but none can predict all aspects of the financial and macroeconomic data. In fact, our analytical results reveal that one pair of commonly assumed preferences and production technologies guarantees inconsistency of at least two aspects of the data. With a CES production function and log–log utility, excess stock market volatility over consumption volatility appears to go hand in hand with a negative comovement of consumption and labor. Exactly the same can be said of a CES production function with log-power utility, which includes as a special case the log-linear utility employed by Hansen (1985) to capture indivisibility of labor. Our model identifies cases in which labor explicitly acts to resolve the puzzles of the financial data considered alone, perhaps hinting at directions to explain all aspects of the data. Possibilities include generalizing the utility and production functions further or allowing for market incompleteness. A related body of literature (Cuoco, 1997; Detemple, 1996; Duffie et al., 1997; Svensson and Werner, 1993, among others) has applied incomplete markets analysis to situations with human capital, but absent an explicit labor choice.

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Appendix A. Proofs

For expositional simplicity, we omit the 't' argument of stochastic processes where unambiguous.

Proof of Proposition 1. We apply Itô's Lemma to Eq. (15) and use the geometric Brownian motion state variable assumption to take the derivative of the expectation with respect to $\varepsilon(t)$, via

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon(t)}\mathrm{E}[g(\varepsilon(s))|\mathscr{F}_t] = \mathrm{E}\left[\frac{\varepsilon(s)}{\varepsilon(t)}g'(\varepsilon(s))|\mathscr{F}_t\right].$$

Accordingly, we derive

$$\frac{L(t)(\sigma_L(t)-\sigma_c(t))}{\sigma_c(t)} = c(t)(1-\beta)\bar{\ell}\,\mathrm{E}\left[\int_t^T \frac{\ell(s)}{(\bar{\ell}-\ell(s))^2}\frac{\sigma_\ell(s)}{\sigma_c(t)}\mathrm{d}s|\mathscr{F}_t\right], \tag{A.1}$$

while Eq. (15) and $W(t) = \beta(T-t)c(t)$ imply

$$W(t) - L(t) = c(t)\,\mathrm{E}\left[\int_t^T \frac{\beta\bar{\ell}-\ell(s)}{\bar{\ell}-\ell(s)}\mathrm{d}s|\mathscr{F}_t\right]. \tag{A.2}$$

By substitution of Eqs. (A.1) and (A.2) into Eq. (14) we deduce (a)–(c). Q.E.D.

Proof of Proposition 2. Inspection of Eqs. (9) and (12) immediately yields Eq. (24). Since $\lim_{\ell\to 0}u_c(f(\ell,\varepsilon),\bar{\ell}-\ell)f_\ell(\ell,\varepsilon) = \infty > \lim_{\ell\to 0}u_h(f(\ell,\varepsilon),\bar{\ell}-\ell)$, and $\lim_{\ell\to\bar{\ell}}u_h(f(\ell,\varepsilon),\bar{\ell}-\ell) = \infty > \lim_{\ell\to\bar{\ell}}u_c(f(\ell,\varepsilon),\bar{\ell}-\ell)f_\ell(\ell,\varepsilon)$, and since $u_c f_\ell$ is decreasing in ℓ while u_h is increasing in ℓ, there exists a unique solution in $(0,\bar{\ell})$ to Eq. (24). Then Eq. (23) yields Eqs. (25) and (26). Eqs. (7) and (8) yield Eqs. (27) and (28), since Eq. (6) holding with equality at equilibrium implies y can be set equal to 1. Applying Itô's Lemma to Eq. (27) and equating drift terms yields Eq. (29). Equating S to F and substituting Eqs. (24), (27) and (28) into Eq. (10) yields Eq. (30). Continuity of the information filtration $\{\mathscr{F}_t\}$, preferences, and production function and the fact that ε is an Itô process imply ℓ is an Itô process via an application of Itô's Lemma to Eq. (24). Subsequently, Itô's Lemma applied to Eqs. (25)–(28) implies c, δ, ξ, and w are Itô processes, while the present value formula Eq. (30) also has an Itô representation. Q.E.D.

Proof of Proposition 3. Applying Itô's Lemma to Eq. (24) yields Eq. (31). Use of Eq. (31) and applying Itô's Lemma to $c = f$, $w = f_\ell$, $\delta = f - f_\ell \ell$ and $\xi = u_c$ yields, respectively, Eqs. (32)–(35). Q.E.D.

Proof of Corollary 1. Using Eqs. (31) and (32) and the fact that for a strictly quasi-concave utility function $u_h^2 u_{cc} + u_c^2 u_{hh} - 2u_c u_h u_{ch} < 0$, we derive that when $f_{\ell\varepsilon} f / f_\ell f_\varepsilon > 1/e_{c,h}$ and $f_\varepsilon > 0$ then $\sigma_\ell/\sigma_\varepsilon > 0$ and $\sigma_c/\sigma_\varepsilon > 0$. Similarly when $f_{\ell\varepsilon} f / f_\ell f_\varepsilon > 1/e_{c,h}$ and $f_\varepsilon < 0$, then $\sigma_\ell/\sigma_\varepsilon < 0$ and $\sigma_c/\sigma_\varepsilon < 0$. For the second part, when $f_{\ell\varepsilon} f / f_\ell f_\varepsilon < 1/e_{c,h}$ and $f_\varepsilon > 0$, then $\sigma_\ell/\sigma_\varepsilon < 0$, and when $f_{\ell\varepsilon} f / f_\ell f_\varepsilon < 1/e_{c,h}$ and $f_\varepsilon < 0$ then $\sigma_\ell/\sigma_\varepsilon > 0$. We compare Eqs. (31) and (32) to observe

$$\frac{\sigma_c}{\sigma_\varepsilon} = \varepsilon \frac{f_\varepsilon}{f} + \frac{f_\ell}{f} \ell \frac{\sigma_\ell}{\sigma_\varepsilon}. \tag{A.3}$$

Then by some (tedious) algebra, we write Eq. (32) as

$$\ell \frac{\sigma_\ell}{\sigma_\varepsilon} = -\frac{1}{\left\{ \dfrac{1}{e_{c,h}} - \dfrac{u_c}{u_h} u_{hh} - \dfrac{u_c^2}{u_h^2} f_{\ell\ell} \right\}} \frac{\varepsilon f_\varepsilon}{f_\ell} \left\{ \frac{1}{e_{c,h}} - \frac{f_{\ell\varepsilon} f}{f_\ell f_\varepsilon} \right\}$$

$$\begin{cases} > \dfrac{-\varepsilon f_\varepsilon}{f_\ell} & \text{if } f_\varepsilon > 0 \text{ and } \dfrac{1}{e_{c,h}} > \dfrac{f_{\ell\varepsilon} f}{f_\ell f_\varepsilon} \geq 0, \\ < \dfrac{-\varepsilon f_\varepsilon}{f_\ell} & \text{if } f_\varepsilon < 0 \text{ and } \dfrac{1}{e_{c,h}} > \dfrac{f_{\ell\varepsilon} f}{f_\ell f_\varepsilon} \geq 0. \end{cases} \tag{A.4}$$

Hence, if $f_\varepsilon > 0$, $0 \leq f_{\ell\varepsilon} f / f_\ell f_\varepsilon < 1/e_{c,h}$, $\sigma_c/\sigma_\varepsilon > \varepsilon f_\varepsilon / f - \varepsilon f_\varepsilon / f = 0$; and if $f_\varepsilon < 0$, $0 \leq f_{\ell\varepsilon} f / f_\ell f_\varepsilon < 1/e_{c,h}$, $\sigma_c/\sigma_\varepsilon < \varepsilon f_\varepsilon / f - \varepsilon f_\varepsilon / f = 0$. Q.E.D.

Proof of Proposition 4. Claims (a)–(e) are clearly equivalent, from Eqs. (31)–(34). Then (a) and Eq. (35) imply (f). The comparisons between $|\sigma_w|$ and $|\sigma_c|$ come directly from (c) and (d), making use of the fact that $1/e_{c,h} > 0$ implies $f_{\ell\varepsilon} f / f_\ell f_\varepsilon > 0$, so either $f_{\ell\varepsilon} > 0, f_\varepsilon > 0$ or $f_{\ell\varepsilon} < 0, f_\varepsilon < 0$. (e) may be rearranged as

$$\sigma_\delta = \frac{f_{\ell\varepsilon}}{f_\ell} \sigma_\varepsilon \varepsilon \frac{(f_\varepsilon / f_{\ell\varepsilon} - \ell)}{(f/f_\ell - \ell)} = \sigma_w \frac{(e_{c,h} f / f_\ell - \ell)}{(f/f_\ell - \ell)}, \tag{A.5}$$

implying

$$|\sigma_\delta| < |\sigma_w| \quad \text{if and only if} \quad -1 < \frac{(e_{c,h} f/f_\ell - \ell)}{(f/f_\ell - \ell)} < 1$$

$$\text{if and only if} \quad -\frac{f}{f_\ell} + \ell < e_{c,h}\frac{f}{f_\ell} - \ell < \frac{f}{f_\ell} - \ell.$$

If $e_{c,h} > 1$, then $e_{c,h} f/f_\ell - \ell > f/f_\ell - \ell$, implying $|\sigma_\delta| > |\sigma_w|$. If $e_{c,h} < 1$, then $e_{c,h} f/f_\ell - \ell < f/f_\ell - \ell$, so we only need to check the left hand inequality. We deduce

$$|\sigma_\delta| < |\sigma_w| \quad \text{if and only if} \quad \frac{f}{f_\ell \ell} > \frac{2}{1 + e_{c,h}}.$$

(e) can also be rearranged as

$$\sigma_\delta = \sigma_c \frac{\left(1 - \dfrac{1}{e_{c,h}} \dfrac{\ell f_\ell}{f}\right)}{\left(1 - \dfrac{\ell f_\ell}{f}\right)} \tag{A.6}$$

implying

$$|\sigma_\delta| < |\sigma_c| \quad \text{if and only if} \quad -1 + \frac{\ell f_\ell}{f} < 1 - \frac{1}{e_{c,h}} \frac{\ell f_\ell}{f} < 1 - \frac{\ell f_\ell}{f}.$$

If $e_{c,h} > 1$, then $-1/e_{c,h} > -1$, so $|\sigma_\delta| > |\sigma_c|$. If $e_{c,h} < 1$, then the right-hand inequality holds, so we only need to check the left-hand one. We deduce

$$|\sigma_\delta| < |\sigma_c| \quad \text{if and only if} \quad \frac{1 + e_{c,h}}{2e_{c,h}} < \frac{f}{\ell f_\ell}.$$

Combining these comparisons, we deduce the remaining results of Proposition 4. Q.E.D.

Proof of Proposition 5. Assuming ε follows a geometric Brownian motion, we may express σ_W and σ_L in terms of σ_c process only:

$$\frac{\sigma_W(t)}{\sigma_c(t)} = \frac{1}{W(t)u_c(t)} \mathrm{E}\left[\int_t^T u_c(s)c(s)\left[1 + \frac{u_{cc}(s)}{u_c(s)} c(s)\right] \frac{\sigma_c(s)}{\sigma_c(t)} \mathrm{d}s \,|\, \mathscr{F}_t\right] - \frac{u_{cc(t)}}{u_c(t)} c(t), \tag{A.7}$$

$$\frac{\sigma_L(t)}{\sigma_c(t)} = \frac{1}{L(t)u_c(t)} \mathrm{E}\left[\int_t^T \ell(s)\, c(s)\, u_h(s) \frac{\sigma_c(s)}{\sigma_c(t)} u_{ch}(s)\, \mathrm{d}s | \mathscr{F}_t\right] - \frac{u_{cc(t)}}{u_c(t)} c(t). \tag{A.8}$$

$\sigma_c(s)/\sigma_c(t) > 0$, $-u_{cc}c/u_c < 1$ and $u_{ch} < 0$ imply from Eqs. (A.7) and (A.8) that $\sigma_W/\sigma_c > -u_{cc}c/u_c$ and $\sigma_L/\sigma_c < -u_{cc}c/u_c$. Hence $\sigma_L/\sigma_W < 1$, which from Eq. (13) implies the desired result. Q.E.D.

Proof of Proposition 6. We derive $f_{\ell\varepsilon}f/f_\ell f_\varepsilon = g'\varepsilon g/\varepsilon g' g = 1$, and $e_{c,h} = 1$. Hence from Proposition 4 we have the results on σ_ℓ, σ_c, σ_w and σ_δ. From Proposition 4(f), we calculate $\theta(t) = (1-\gamma\beta)\sigma_c(t)$. Hence from Eq. (A.7),

$$\sigma_W(t) = \frac{\sigma_\varepsilon}{W(t)\,u_c(t)} \mathrm{E}\left[\int_t^T \gamma\beta u_c(s)\,c(s)\,\mathrm{d}s|\mathscr{F}_t\right] + (1-\gamma\beta)\sigma_\varepsilon = \sigma_\varepsilon, \tag{A.9}$$

and from Eq. (A.8)

$$\begin{aligned}\sigma_L(t) &= \frac{\sigma_\varepsilon \ell(t)}{L(t)\,u_c(t)} \mathrm{E}\left[\int_t^T \gamma\beta(1-\gamma)c(s)^{\gamma\beta}\,h(s)^{\gamma(1-\beta)-1}\,\mathrm{d}s|\mathscr{F}_t\right] + (1-\gamma\beta)\sigma_\varepsilon \\ &= \frac{\sigma_\varepsilon}{L(t)\,u_c(t)}\gamma\beta\,\mathrm{E}\left[\int_t^T \ell(s)\,u_h(s)\,\mathrm{d}s|\mathscr{F}_t\right] + (1-\gamma\beta)\sigma_\varepsilon \\ &= \sigma_\varepsilon = \sigma_W(t).\end{aligned} \tag{A.10}$$

Hence, from Eq. (12) $\sigma(t) = \sigma_\varepsilon$, as required. Q.E.D.

Proof of Proposition 7. (a) For $v > \alpha > 0, f_\varepsilon > 0, f_{\ell\varepsilon} < 0$ and $f_{\ell\varepsilon}f/f_\ell f_\varepsilon < 1 = 1/e_{c,h}$. Hence Eq. (31) yields the result for $\sigma_\ell/\sigma_\varepsilon$. Substituting for log utility and $f_{\ell\varepsilon}f/f_\ell f_\varepsilon = 1 - v/\alpha$ into Eq. (31) and using Eq. (24) we obtain

$$\frac{\sigma_\ell}{\sigma_\varepsilon} = -\frac{\varepsilon}{\ell}\frac{\beta}{\{1/h - \beta f_{\ell\ell}/f_\ell\}}\frac{f_\varepsilon}{f}\frac{v}{\alpha}. \tag{A.11}$$

Then from Eq. (32),

$$\begin{aligned}\frac{\sigma_c}{\sigma_\varepsilon} &= \frac{\varepsilon f_\varepsilon}{f} + \frac{f_\ell \ell}{f}\frac{\sigma_\ell}{\sigma_\varepsilon} = \frac{\varepsilon f_\varepsilon}{f}\left\{1 - \frac{f_\ell}{f}\frac{\beta}{\{1/h - \beta f_{\ell\ell}/f_\ell\}}\frac{v}{\alpha}\right\} \\ &= \frac{\varepsilon f_\varepsilon}{f\{1/h - \beta f_{\ell\ell}/f_\ell\}}\left\{\frac{1}{h} - \frac{\beta f_{\ell\ell}}{f_\ell} - \frac{\beta v}{\alpha}\frac{f_\ell}{f}\right\} \\ &= \frac{\varepsilon f_\varepsilon \beta}{f\{1/h - \beta f_{\ell\ell}/f_\ell\}}\left\{\left(\frac{1}{1-\beta} - \frac{v}{\alpha}\right)\frac{f_\ell}{f} - \frac{f_{\ell\ell}}{f_\ell}\right\},\end{aligned} \tag{A.12}$$

where for the last equality we have used Eq. (24) to replace $1/h$ by $\beta f_\ell/(1-\beta)f$. Then since

$$f_{\ell\ell}/f_\ell = \frac{-(1-v)}{\ell} + \frac{(\alpha - v)\eta\ell^{v-1}}{[\eta\ell^v + (1-\eta)\ \varepsilon^v]} \quad \text{and} \quad f_\ell/f = \frac{\alpha\eta\ell^{v-1}}{[\eta\ell^v + (1-\eta)\ \varepsilon^v]},$$

we obtain

$$\frac{\sigma_c}{\sigma_\varepsilon} = \frac{\varepsilon f_\varepsilon \beta}{f\{1/h - \beta f_{\ell\ell}\}} \left\{ \left(\frac{\alpha}{1-\beta} - v - \alpha + v \right) \frac{\eta \ell^{v-1}}{[\eta \ell^v + (1-\eta)\varepsilon^v]} + \frac{(1-v)}{\ell} \right\} > 0 \tag{A.13}$$

as required. Also $\sigma_\ell(s)/\sigma_c(t) < 0$, as required since σ_ε is a constant. Eq. (33) implies

$$\frac{\sigma_w}{\sigma_\varepsilon} > \frac{\varepsilon f_{\ell\varepsilon}}{f_\ell} = \frac{-(v-\alpha)(1-\eta)\varepsilon^v}{[\eta\ell^v + (1-\eta)\varepsilon^v]} > -(v-\alpha).$$

The result $|\sigma| > |\sigma_c|$ comes from Proposition 1 and the fact that $\sigma_\ell(s)/\sigma_c(t) < 0$.

(b) For $\alpha \geq v > 0, f_\varepsilon > 0, f_{\ell\varepsilon} \geq 0$ and $f_{\ell\varepsilon} f / f_\ell f_\varepsilon < 1 = 1/e_{c,h}$. Hence Eq. (31) yields the result for $\sigma_\ell/\sigma_\varepsilon$, then Corollary 1 yields the result for $\sigma_c/\sigma_\varepsilon$ and Eq. (33) yields $\sigma_w/\sigma_\varepsilon > 0$. Again the result $|\sigma| > |\sigma_c|$ comes from Proposition 1. Using $f_{\ell\varepsilon}/f_\ell = (-v/\alpha) f_\varepsilon/f$ and manipulating Eq. (33) yields

$$\frac{\sigma_w}{\sigma_\varepsilon} = \frac{\varepsilon f_\varepsilon}{f}\left[1 - \frac{v}{\alpha}\left(1 + \frac{f_{\ell\ell}}{f_\ell} V\right)\right], \tag{A.14}$$

which combined with Eq. (A.12) yields

$$\frac{\sigma_c}{\sigma_\varepsilon} - \frac{\sigma_w}{\sigma_\varepsilon} = \frac{\varepsilon f_\varepsilon}{f}\frac{v}{\alpha}\left[\frac{(1-\gamma)/h}{\left(\frac{1-\beta}{\beta}\right)\frac{1}{h^{1-\gamma}} + \frac{1-\gamma}{h} - \frac{f_{\ell\ell}}{f}}\right], \tag{A.15}$$

implying $|\sigma_w| < |\sigma_c|$.

(c) For $v < 0$, $f_\varepsilon > 0$, $f_{\ell\varepsilon} \geq 0$ and $f_{\ell\varepsilon} f / f_\ell f_\varepsilon > 1 = 1/e_{c,h}$. Hence Eq. (31) yields $\sigma_\ell/\sigma_\varepsilon > 0$, and Eq. (32) yields $\sigma_c/\sigma_\varepsilon > 0$. Then Eq. (33) implies

$$\sigma_w/\sigma_\varepsilon < \varepsilon f_{\ell\varepsilon}/f_\ell = \frac{(\alpha - v)(1-\eta)\varepsilon^v}{[\eta\ell^v + (1-\eta)\varepsilon^v]} < \alpha - v.$$

Eq. (A.14) yields $\sigma_w > 0$ and Eq. (A.15) the $|\sigma_w|$, $|\sigma_c|$ comparison.

All the results on θ in (a), (b), (c) follow since $\theta = \sigma_c$ for log utility. Q.E.D.

Proof of Proposition 8. Since $v = \alpha > 0$, we have from Proposition 7, the $\sigma_\ell/\sigma_\varepsilon$, $\sigma_c/\sigma_\varepsilon$, $\sigma_w/\sigma_\varepsilon$ and $\theta/\sigma_\varepsilon$ results. Since $f_{\ell\varepsilon} = 0$, from Eqs. (31) and (33) we have

$$\frac{\sigma_w}{\sigma_\varepsilon} = \frac{f_{\ell\ell}\ell}{f_\ell}\frac{\sigma_\ell}{\sigma_\varepsilon} = -(1-v)\frac{\sigma_\ell}{\sigma_\varepsilon} < -\frac{\sigma_\ell}{\sigma_\varepsilon} \tag{A.16}$$

implying the $|\sigma_\ell|$ and $|\sigma_w|$ comparison. Substituting for log utility and $f_{\ell\varepsilon}f/f_\ell f_\varepsilon = 0$, into Eq. (31) we derive

$$\frac{\sigma_\ell}{\sigma_\varepsilon} = -\frac{\varepsilon}{\ell}\frac{\beta}{\left\{\dfrac{1}{h} - \dfrac{\beta^2 h}{(1-\beta)}\dfrac{f_{\ell\ell}}{f}\right\}}\frac{f_\varepsilon}{f} = -\frac{\varepsilon}{\ell}\frac{\beta}{\left\{\dfrac{1}{h} - \dfrac{\beta f_{\ell\ell}}{f_\ell}\right\}}\frac{f_\varepsilon}{f}, \tag{A.17}$$

making use of Eq. (24) for log utility. Then from Eq. (32),

$$\frac{\sigma_c}{\sigma_\varepsilon} = \frac{\varepsilon f_\varepsilon}{f} + \frac{f_\ell \ell}{f}\frac{\sigma_\ell}{\sigma_\varepsilon} = \frac{\varepsilon f_\varepsilon}{f}\left\{1 - \frac{f_\ell}{f}\frac{\beta}{\left\{\dfrac{1}{h} - \dfrac{\beta f_{\ell\ell}}{f_\ell}\right\}}\right\}. \tag{A.18}$$

Hence

$$\frac{\sigma_c}{\sigma_w} = \frac{\dfrac{\beta f_\ell \ell}{(1-\beta)f} + \beta(1-v) - \dfrac{\beta f_\ell \ell}{f}}{\beta(1-v)} > 1. \tag{A.19}$$

Hence, we have $|\sigma_c| > |\sigma_w|$, as required. We have $\sigma_\ell(s)/\sigma_c(t) < 0$, implying $|\sigma| > |\sigma_c|$ from Proposition 1. Q.E.D.

Proof of Proposition 9. We have $f_{\ell\varepsilon}f/f_\ell f_\varepsilon = 1$ and $1/e_{c,h} = 1 - \rho$. Substituting for this utility function into Eq. (31) we obtain

$$\frac{\sigma_\ell}{\sigma_\varepsilon} = \frac{\rho/\ell}{\left\{\dfrac{1-\rho}{h} + \dfrac{(1-\beta)(1-\rho)}{\beta c^\rho h^{1-\rho}} - \dfrac{\beta}{1-\beta}\dfrac{h^{1-\rho}}{c^{1-\rho}} f_{\ell\ell}\right\}} > 0 \quad \text{if and only if} \quad \rho > 0, \tag{A.20}$$

yielding the first result in parts (a) and (b). Substituting into Eq. (32), and using $f_\varepsilon = f/\varepsilon$, we obtain

$$\frac{\sigma_c}{\sigma_\varepsilon} = 1 + \frac{\rho f_\ell/f}{\left\{\dfrac{1-\rho}{h} + \dfrac{(1-\beta)(1-\rho)}{\beta c^\rho h^{1-\rho}} - \dfrac{\beta}{1-\beta}\dfrac{h^{1-\rho}}{c^{1-\rho}} f_{\ell\ell}\right\}} > 1 \quad \text{if and only if} \quad \rho > 0. \tag{A.21}$$

Substituting for $c = f$ and

$$f_\ell = \frac{1-\beta}{\beta}\left(\frac{c}{h}\right)^{1-\rho},$$

from Eq. (24), we can rearrange as

$$\frac{\sigma_c}{\sigma_\varepsilon} = 1 + \frac{\rho}{\left\{\frac{\beta}{1-\beta}(1-\rho)\left(\frac{c}{h}\right)^{\rho} + (1-\rho) - \frac{\beta^2}{1-\beta}\left(\frac{h}{c}\right)^{2-2\rho} f_{\ell\ell}\right\}} \begin{cases} < 1/(1-\rho) & \text{if } \rho > 0 \\ > 1/(1-\rho) & \text{if } \rho < 0 \end{cases} \tag{A.22}$$

yielding the second and third result in parts (a) and (b).

Substituting for CES $u(c,h)$ into Eq. (33) and using $f_{\ell\varepsilon} = f_\ell/\varepsilon$, we obtain

$$\frac{\sigma_w}{\sigma_\varepsilon} = 1 + \frac{\rho f_{\ell\ell}/f_\ell}{\left\{\frac{1-\rho}{h} + \frac{(1-\beta)(1-\rho)}{\beta c^{\rho} h^{1-\rho}} - \frac{\beta}{1-\beta}\frac{h^{1-\rho}}{c^{1-\rho}} f_{\ell\ell}\right\}} > 1 \quad \text{if and only if} \quad \rho < 0. \tag{A.23}$$

Substituting for

$$f_\ell = \frac{1-\beta}{\beta}\left(\frac{c}{h}\right)^{1-\rho}$$

we can rearrange as

$$\frac{\sigma_w}{\sigma_\varepsilon} = 1 - \frac{\rho}{\left\{1 - \frac{(1-\beta)}{\beta}\frac{(1-\rho)}{f_{\ell\ell}}\frac{c^{1-\rho}}{h^{2-\rho}} - \frac{(1-\beta)(1-\rho)c^{1-2\rho}}{\beta^2 h^{2-2\rho} f_{\ell\ell}}\right\}} \begin{cases} > 1-\rho & \text{if } \rho > 0, \\ < 1-\rho & \text{if } \rho < 0, \end{cases} \tag{A.24}$$

yielding the fourth results in parts (a) and (b). Substituting for CES $u(c,h)$ in Eq. (34) and using $f_\varepsilon - \ell f_{\ell\varepsilon} = (f - \ell f_\ell)/\varepsilon$, we obtain

$$\frac{\sigma_\delta}{\sigma_\varepsilon} = 1 - \frac{\rho \ell f_{\ell\ell}/(f - \ell f_\ell)}{\left\{\frac{1-\rho}{h} + \frac{(1-\beta)(1-\rho)}{\beta c^{\rho} h^{1-\rho}} - \frac{\beta}{1-\beta}\frac{h^{1-\rho}}{c^{1-\rho}} f_{\ell\ell}\right\}} > 1 \quad \text{if and only if} \quad \rho > 0. \tag{A.25}$$

The $|\sigma_c|$ and $|\sigma_w|$ comparisons are immediate from the previous bounds on $\sigma_c/\sigma_\varepsilon$ and $\sigma_w/\sigma_\varepsilon$. A comparison of Eqs. (A.21) and (A.25) delivers the $|\sigma_c|$ and $|\sigma_\delta|$ comparisons. Q.E.D.

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