An Equilibrium Model with Restricted Stock Market Participation

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This article solves the equilibrium problem in a pure-exchange, continuous-time economy in which some agents face information costs or other types of frictions effectively preventing them from investing in the stock market. Under the assumption that the restricted agents have logarithmic utilities, a complete characterization of equilibrium prices and consumption/ investment policies is provided. A simple calibration shows that the model can help resolve some of the empirical asset pricing puzzles.

It is well documented that even in well-developed capital markets, a large fraction of households does not participate in the stock market. For example, Mankiw and Zeldes (1991) report that 72.4% of the households in a representative sample from the *1984 Panel Study of Income Dynamics* held no stocks at all.[1] These households earned 62% of the aggregate disposable income and accounted for 68% of aggre-

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[1] This figure does not take into account indirect stock ownership through pension funds. Based on their analysis of the *1989 Survey of Consumer Finances*, Blume and Zeldes (1994) report that only 32.7% of households held stock directly or indirectly through mutual funds, IRAs, trusts, or defined-contribution pension plans, and only 19.4% had direct or indirect stock holdings in excess of $5,000.

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gate food expenditures. Even more surprisingly, only 47.7% of the households holding other liquid assets in excess of $100,000 held any equity. The fraction of households owning stocks increases with average labor income and education, thus lending some support to the presence of fixed information costs. Mankiw and Zeldes also document a significant difference in the consumption patterns of stockholders versus nonstockholders: in particular, the aggregate consumption of stockholders is more highly correlated with the equity risk premium than is the aggregate consumption of nonstockholders, and is more volatile. As a result, Mankiw and Zeldes found that estimating the consumption-based CAPM of Breeden (1979) using only the aggregate consumption of stockholders results in a lower estimate of the representative risk aversion coefficient. Thus restricted stock market participation might help resolve the documented inability of frictionless equilibrium models to account for the size of the average spread between equity returns and the short-term interest rate, at least for "reasonable" levels of representative risk aversion. This inability was first noticed by Mehra and Prescott (1985), and has been referred to as the "equity premium puzzle."[2]

This article characterizes the equilibrium in a pure-exchange, continuous-time economy in which a fraction of the population is prevented from investing in the stock market because of information costs or other types of frictions. The restricted agents (whose preferences are taken to be logarithmic throughout) may only invest in a locally riskless (overnight) bond market, and thus face an incomplete market. The remaining unrestricted agents, on the other hand, also have access to a stock which is a claim to an exogenously given dividend process. The bond is assumed to be in zero net supply, while the stock is in positive net supply. The consumption allocation of the restricted and unrestricted agents, as well as their optimal bond holdings and the equilibrium interest rate and stock price process are determined endogenously.

The construction of the equilibrium is achieved by introducing a representative agent's utility function assigning stochastic weights to the two classes of agents in the economy. This allows us to account for the different investment opportunities faced by the two agents and to easily impose the optimality and market-clearing conditions. In equilibrium, the stochastic weights act as a proxy for stochastic shifts in

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[2] Constantinides and Duffie (1996) have shown that the "equity premium puzzle" could be resolved in an incomplete market setting in which agents face uninsurable income shocks. However, Heaton and Lucas (1996) have provided some evidence suggesting that the actual size and persistence of income shifts might be insufficient to generate realistic values for the equity premium. Alternative explanations have been based on non-time-additive preferences or market frictions [see, e.g., Constantinides (1990), Aiyagari and Gertler (1991), and Heaton and Lucas (1996)].

the distribution of wealth between the two classes of agents. The weighting process is explicitly characterized and is shown to be negatively correlated with aggregate consumption and to have a volatility that is decreasing in the level of aggregate consumption. This weaker notion of aggregation was introduced—in the context of incomplete markets—by Cuoco and He (1994a,b), who however did not provide a complete characterization of the equilibrium weighting process in terms of exogenous quantities, and has recently been exploited by Serrat (1995) in the presence of borrowing constraints and by Basak (1996b) in the presence of heterogeneous beliefs.

Our analysis reveals that in equilibrium the restricted agents choose a consumption process having no covariation with aggregate consumption (or with the stock price). As a consequence, the unrestricted agents must absorb all the aggregate consumption risk into their consumption stream. Accordingly, the restricted agents' consumption volatility is decreased in comparison to the benchmark economy in which all agents have free access to the stock market, while the unrestricted agents' consumption volatility is increased, in agreement with the empirical finding of Mankiw and Zeldes (1991).

A further consequence of the restricted agents choosing a consumption plan having no covariation with the stock price process is that the equilibrium interest rate directly responds only to the precautionary saving motive (prudence) of the unrestricted agents, while the stock risk premium is proportional to the unrestricted agents' risk aversion (rather than to the representative agent's). A simple calibration reveals that our model can exactly match the historical market price of risk even for a representative agent's relative risk aversion as small as 1.3, while at the same time generating much smaller values for the real interest rate than the standard representative agent model. This confirms Mankiw and Zeldes' intuition that restricted participation might contribute to explaining the asset pricing "puzzles" of Mehra and Prescott (1985) and Weil (1989).[3]

To derive further implications of restricted market participation, we also specialize our setup to the case of both classes of agents having logarithmic preferences and of the aggregate dividend process following a geometric Brownian motion. For this case we show that, as a consequence of restricted market participation, the equilibrium interest rate becomes procyclical and the risk premium anticyclical.

[3] In the specification of the model that we consider for calibration, the equilibrium interest rate and stock risk premium can be expressed as deterministic functions of a single observable state variable (the nonstockholders' share of aggregate consumption). Therefore, differently from Mehra and Prescott (1985) and Weil (1989), we report the equilibrium interest rate and stock risk premium implied by the model conditional on the empirical value of the state variable rather than their expected values under the stationary distribution for the state variable.

However, the stock price is unaffected. In the restricted economy, the volatilities of both the interest rate and the stock risk premium are decreasing in aggregate consumption. Moreover, limited stock market participation unambiguously results in a lower interest rate and a higher risk premium. Finally, we find that removing the restrictions to stock market participation increases the welfare of the nonstockholders, but decreases the welfare of the stockholders.

In closely related work, Saito (1996) examines the implications of restricted market participation in a continuous-time production economy with a single linear technology whose return process follows a geometric Brownian motion. His analytic results are limited to the case of both agents having logarithmic preferences, for which he reports that the interest rate is decreased as a result of restricted stock market participation. Since in such a production economy the moments of the rate of return on the risky asset equal those of the rate of return on the technology (rather than being endogenously determined), this also immediately implies that the equity risk premium is increased. Of course, such an implication would not necessarily hold in a pure-exchange economy, like the one we consider. In addition, such a model is unable to match the historical real interest rate and equity risk premium, leading Saito to also solve numerically a version of the model in which both agents are assumed to have more general Kreps-Porteus preferences with unit elasticity of intertemporal substitution, which result in myopic consumption policies.

Other theoretical models incorporating agents with restricted access to the stock market exist and pursue a variety of objectives, but are limited to one- or two-period settings. Merton (1987) considers a single-period mean-variance setting in which some investors do not invest in all the traded stocks because of incomplete information. As a consequence, the CAPM does not characterize equilibrium risk premia, and the firm's size, the stock's idiosyncratic risk and the stock's investor base emerge as additional factors in explaining returns. Hirshleifer (1988) analyzes a similar model in which participation in the futures market is limited by the existence of a fixed setup cost, and the investor's decision to enter the market is endogenized. Cuny (1993) examines the optimal design of futures contracts, assuming that investors can only participate in one market. Allen and Gale (1994) study a two-period model in which agents face short-sale constraints, a fixed setup cost of participating in the stock market, and liquidity (preference) shocks that might force them to consume at the end of the first period rather than at the end of the second period. They show that limited market participation can amplify the effect of liquidity trading relative to full participation and can thus increase the stock market volatility. Moreover, the model admits two different equilibria with different participation regimes and different stock volatility. These equilibria can be Pareto ranked, with the Pareto preferred equilibrium being characterized by greater participation and lower volatility. Balasko, Cass, and Siconolfi (1990) and Balasko, Cass, and Shell (1995) study the existence and multiplicity of equilibria, with or without "sunspots," in markets with restricted participation.

Closely related to the analysis of limited stock market participation is the literature studying the effects of international capital market segmentation on asset prices, portfolio choices, and welfare. In a single-period mean-variance framework with exogenously fixed interest rates, Black (1974), Subrahmanyam (1975), Stulz (1981), Errunza and Losq (1985, 1989), and Eun and Janakiramanan (1986) show that segmentation increases the segmented stock market risk premium and results in a decrease in welfare for all countries. Again in a single-period mean-variance framework, but with an endogenously determined interest rate, Basak (1996a) shows that international segmentation results in a lower interest rate, as compared to a fully integrated worldwide capital market, while the impact of segmentation on risk premia and welfares is ambiguous. Sellin and Werner (1993) consider an international version of the continuous-time production economy of Cox, Ingersoll, and Ross (1985) in which investors face a binding constraint on the fraction of the foreign capital stock they are allowed to own. Under the assumption of logarithmic utilities for both the domestic and the foreign representative investor, they show that such a constraint results in a lower interest rate for international borrowing and lending.

The rest of the article is organized as follows. The economic setup is described in Section 1. For purpose of comparison, we start our analysis in Section 2 by briefly recalling the standard construction of an equilibrium for the unrestricted case in which both agents have access to the stock market. In Section 3 we provide the main results on the existence and characterization of equilibria in the restricted case. In Section 4 we obtain some additional results for the special case in which both agents have logarithmic utilities. Section 5 provides some concluding remarks, and the Appendix contains all the proofs.

1. The Economy

We consider a continuous-time economy on the finite time span $[0, T]$, modeled as follows.

1.1 Information structure

The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P})$, on which is defined a one-dimensional Brownian motion w. The filtration $\mathbf{F} = \{\mathcal{F}_t\}$ is the augmentation under $\mathcal{P}$ of the filtration generated by w. We assume that $\mathcal{F} = \mathcal{F}_T$, or that the true state of nature is completely determined by the sample paths of w on $[0, T]$. We interpret the sigma-field $\mathcal{F}_t$ as representing the information available at time t and the probability measure $\mathcal{P}$ as representing the agents' common beliefs. All the stochastic processes to appear in the sequel are progressively measurable with respect to $\mathbf{F}$ and all the equalities involving random variables are understood to hold $\mathcal{P}$-a.s.

1.2 Consumption space

There is a single perishable good (the numeraire). The agents' consumption set $\mathcal{C}$ is given by the set of nonnegative progressively measurable consumption rate processes c with $\int_0^T |c(t)|\,dt < \infty$.

1.3 Securities market

The investment opportunities are represented by a locally riskless bond earning the instantaneous interest rate r and a risky stock representing a claim to an exogenously given strictly positive dividend process δ, with

$$\delta(t) = \delta(0) + \int_0^t \mu_\delta(s)\,ds + \int_0^t \sigma_\delta(s)\,dw(s), \tag{1}$$

where μ_δ and $\sigma_\delta > 0$ are arbitrary progressively measurable processes. The initial bond value is normalized to unity so that the bond price process is given by

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right). \tag{2}$$

Moreover, it will be shown that in equilibrium the stock price S follows an Itô process:

$$S(t) = S(0) + \int_0^t (S(s)\mu(s) - \delta(s))\,ds + \int_0^t S(s)\sigma(s)\,dw(s). \tag{3}$$

The interest rate process r and the stock price process S (and hence the coefficients μ and σ) are to be determined endogenously in equilibrium.

1.4 Trading strategies

Trading takes place continuously. An admissible trading strategy is a two-dimensional vector process (α, θ), where $\alpha(t)$ and $\theta(t)$ denote the amounts invested at time t in the bond and in the stock, respectively, satisfying

$$\int_0^T |\alpha(t)r(t) + \theta(t)\mu(t)|\, dt + \int_0^T |\theta(t)\sigma(t)|^2\, dt \ < \ \infty$$

and

$$\alpha(t) + \theta(t) \geq 0 \qquad \forall t \in [0, T].$$

The set of admissible trading strategies is denoted by Θ.

A trading strategy $(\alpha, \theta) \in \Theta$ is said to *finance* the consumption plan $c \in \mathcal{C}$ if the corresponding wealth process $W = \alpha + \theta$ satisfies the dynamic budget constraint

$$dW(t) = (\alpha(t)r(t) + \theta(t)\mu(t) - c(t))\, dt + \theta(t)\sigma(t)\, dw(t). \qquad (4)$$

1.5 Agents' preferences and endowments

The economy is populated by two agents (or classes of agents). The first agent has access to both the bond and the stock market, while the second is prevented from investing in the stock market.[4] Preferences for agent i $(i = 1, 2)$ are represented by a time-additive von Neumann-Morgenstern utility function

$$U_i(c) = \mathrm{E}\left[\int_0^T e^{-\rho t} u_i(c(t))\, dt\right]$$

for some $\rho > 0$. We assume throughout this article that $u_2(c) = \log(c)$ and that the function $u_1 : (0, \infty) \to \mathbb{R}$ is increasing, strictly concave, and three times continuously differentiable. Moreover, u_1 satisfies the Inada conditions

$$\lim_{c\downarrow 0} u_1'(c) = \infty \quad \text{and} \quad \lim_{c\uparrow\infty} u_1'(c) = 0. \qquad (5)$$

The special case in which $u_1(c) = u_2(c) = \log c$ is considered in Section 4.

Remark 1. *Equation (5) is well understood and it implies that the nonnegativity constraint on consumption is not binding and that the derivative function u_1' has a continuous and strictly decreasing inverse f_1 mapping $(0, \infty)$ onto itself.*

Agent 1 is endowed with one share of the stock and a short position in β shares of the bond, while agent 2 is endowed with β shares of

[4] Rather than introducing opportunity costs explicitly, we treat the constraint as exogenous. Saito (1996) provides some calibration evidence that a relatively small opportunity cost (of the order of 1 to 2 hours of work per week) may deter the bottom 90% of households or one-third of the aggregate wealth from participating in the stock market.

the bond. Thus the supply of the stock is normalized to one share, while the bond is in zero net supply. We assume that

$$\beta < \frac{1 - e^{-\rho T}}{\rho}\,\delta(0), \tag{6}$$

which essentially restricts agent 1 to not be so deeply in debt at the initial date that he can never pay back from the dividend supply. It will be shown later that this condition is necessary to guarantee the existence of an equilibrium.

1.6 Equilibrium

We will denote by $\mathcal{E} = ((\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P}), \delta, U_1, U_2, \beta)$ the primitives for the above economy. An *equilibrium* for the economy $\mathcal{E}$ is a price process (B, S)—or equivalently an interest rate stock price process (r, S)—and a set $\{c_i^*, (\alpha_i^*, \theta_i^*)\}_{i=1}^2$ of consumption and admissible trading strategies for the two agents such that

(i) (α_i^*, θ_i^*) finances c_i^* for $i = 1, 2$;
(ii) c_1^* maximizes U_1 over the set of consumption plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$ with $\alpha(0) + \theta(0) = S(0) - \beta$;
(iii) c_2^* maximizes U_2 over the set of consumption plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$ with $\alpha(0) + \theta(0) = \beta$ and $\theta \equiv 0$; and
(iv) all markets clear, that is, $c_1^* + c_2^* = \delta$, $\alpha_1^* + \alpha_2^* \equiv 0$ and $\theta_1^* = S$.

We will also sometimes refer to an equilibrium as a quadruple (B, S, c_1^*, c_2^*), or (r, S, c_1^*, c_2^*), without explicit reference to the associated trading strategies.

2. The Unrestricted Case

If the second agent had unrestricted access to the stock market, an equilibrium for the above economy could be constructed as in Huang (1987) or Karatzas, Lehoczky, and Shreve (1990) by replacing the two agents with a single representative agent endowed with the aggregate supply of securities and with utility function

$$U(c; \lambda) = \mathrm{E}\left[\int_0^T e^{-\rho t} u(c(t), \lambda)\, dt\right],$$

where

$$u(c, \lambda) = \max_{c_1 + c_2 = c} u_1(c_1) + \lambda u_2(c_2) \tag{7}$$

for some $\lambda > 0$. In order to facilitate the comparison with the restricted case and to motivate the developments in the next section, we briefly recall this construction.

Since consuming the aggregate dividend must be optimal for the representative agent, the marginal rate of substitution process

$$\xi(t) = e^{-\rho t} \frac{u_c(\delta(t), \lambda)}{u_c(\delta(0), \lambda)}$$

identifies the equilibrium state price density, so that the dynamics of ξ are given by

$$d\xi(t) = -\xi(t) r(t)\, dt - \xi(t)\kappa(t)\, dw(t), \tag{8}$$

where κ denotes the market price of risk process, that is,

$$\mu(t) - r(t) = \kappa(t)\sigma(t). \tag{9}$$

In turn, this implies the well-known relations

$$r(t) = -\frac{\mathcal{D}(e^{-\rho t} u_c(\delta(t), \lambda))}{e^{-\rho t} u_c(\delta(t), \lambda)} \tag{10}$$

and

$$S(t) = \mathrm{E}\left[\int_t^T e^{-\rho(s-t)} \frac{u_c(\delta(s), \lambda)}{u_c(\delta(t), \lambda)} \delta(s)\, ds \,\middle|\, \mathcal{F}_t \right], \tag{11}$$

where $\mathcal{D}(e^{-\rho t} u_c(\delta(t), \lambda))$ denotes the drift of the process $e^{-\rho t} u_c(\delta(t), \lambda)$. Equations (8) through (11) imply that the stock price process S has the Itô representation of Equation (3), and that the equilibrium interest rate and risk premium can be expressed as

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2} A(t)P(t)\sigma_\delta(t)^2 \tag{12}$$

and

$$\mu(t) - r(t) = A(t)\sigma_\delta(t)\sigma(t), \tag{13}$$

where

$$A(t) = -\frac{u_{cc}(\delta(t), \lambda)}{u_c(\delta(t), \lambda)}$$

and

$$P(t) = -\frac{u_{ccc}(\delta(t), \lambda)}{u_{cc}(\delta(t), \lambda)}$$

denote the absolute risk aversion and absolute prudence coefficient at time t for the representative agent. In particular, if the stock volatility σ does not vanish, then Equation (13) uniquely identifies the market price of risk process:

$$\kappa(t) = A(t)\sigma_\delta(t). \tag{14}$$

Now let $c_1^*(t)$ and $c_2^*(t)$ denote the solution of the maximization problem in Equation (7) when $c = \delta(t)$, that is,

$$c_1^*(t) = f_1(u_c(\delta(t), \lambda)) \tag{15}$$

and

$$c_2^*(t) = f_2(u_c(\delta(t), \lambda)/\lambda), \tag{16}$$

where f_i denotes the inverse of the marginal utility function u_i'. Equivalently,

$$u_1'(c_1^*(t)) = \lambda u_2'(c_2^*(t)) = u_c(\delta(t), \lambda).$$

Then the given processes (r, S, c_1^*, c_2^*) give rise to an equilibrium for the economy $\mathcal{E}$ provided that the constant $\lambda > 0$ defining the representative agent's utility function is chosen so that the budget constraint for agent 2 is satisfied, that is,

$$\beta = \mathrm{E}\left[\int_0^T e^{-\rho t} \frac{u_c(\delta(t), \lambda)}{u_c(\delta(0), \lambda)} c_2^*(t)\, dt\right] = \frac{1 - e^{-\rho T}}{\rho} \frac{\lambda}{u_c(\delta(0), \lambda)}, \tag{17}$$

where the second equality follows from the fact that $f_2(y) = y^{-1}$.

Remark 2. *It can be easily verified from the definition of the function u in Equation (7) that the function* $g(\lambda) = \lambda / u_c(\delta(0), \lambda)$ *is a strictly increasing continuous map from* $(0, \infty)$ *onto* $(0, \delta(0))$. *Hence Equation (6) is necessary and sufficient to ensure the existence of a unique strictly positive solution* λ *to Equation (17), and hence the existence of an equilibrium in the unrestricted case.*

It follows from an application of Itô's lemma that the optimal consumption policies satisfy

$$dc_i^*(t) = \mu_{c_i^*}(t)\, dt + \sigma_{c_i^*}(t)\, dw(t)$$

where

$$\mu_{c_i^*}(t) = \frac{A(t)}{A_i(t)}\mu_\delta(t) - \frac{1}{2}\frac{A(t)}{A_i(t)}P(t)\sigma_\delta(t)^2 + \frac{1}{2}\frac{A(t)^2}{A_i(t)^2}P_i(t)\sigma_\delta(t)^2,$$

$$\sigma_{c_i^*}(t) = \frac{A(t)}{A_i(t)}\sigma_\delta(t)$$

and A_i and P_i denote the coefficients of absolute risk aversion and absolute prudence for agent i, that is,

$$A_i(t) = -\frac{u_i''(c_i^*(t))}{u_i'(c_i^*(t)}$$

and

$$P_i(t) = -\frac{u_i'''(c_i^*(t))}{u_i''(c_i^*(t))}.$$

Thus the optimal consumption policies are locally perfectly positively correlated with aggregate consumption.

The wealth process for agent 2 is given by

$$\begin{aligned} W_2(t) &= \mathrm{E}\left[\int_t^T e^{-\rho(s-t)} \frac{u_c(\delta(s),\lambda)}{u_c(\delta(t),\lambda)} c_2^*(s)\, ds \,\middle|\, \mathcal{F}_t\right] \\ &= \beta \frac{1-e^{-\rho(T-t)}}{1-e^{-\rho T}} \frac{u_c(\delta(0),\lambda)}{u_c(\delta(t),\lambda)}, \end{aligned}$$

which implies

$$\alpha_2^*(t) = \left(1 - \frac{\mu(t)-r(t)}{\sigma(t)^2}\right) W_2(t)$$

and

$$\theta_2^*(t) = \frac{\mu(t)-r(t)}{\sigma(t)^2} W_2(t).$$

The corresponding quantities for agent 1 are given by $W_1(t) = S(t) - W_2(t)$, $\alpha_2^*(t) = -\alpha_1^*(t)$ and $\theta_1^*(t) = S(t) - \theta_2^*(t)$.

3. The Restricted Case

When agent 2 is restricted from participating in the stock market, the equilibrium consumption allocation is not Pareto efficient, and hence the usual construction of a representative agent as a linear combination (with constant weights) of the individual utility functions is in general not possible. However, the aggregation of the individual preferences into a representative agent utility function is still possible: the representative agent's utility is a weighted average of the individual investor's utilities, but with weightings represented by a progressively measurable process. This allows us to easily account for the market-clearing conditions and to reduce the search for an equilibrium to the specification of the weighting process, since once this has been determined the optimal policies and the equilibrium prices can be easily recovered from the representative agent problem. This weaker notion of aggregation was introduced by Cuoco and He (1994a,b) in the context of incomplete markets.

Before stating formally our main characterization results (Theorem 1 below), we provide some heuristic discussion under the maintained assumption that the equilibrium stock volatility process σ does not vanish (Theorem 1 shows that this condition is in fact not necessary).

Consider the representative agent with state-dependent utility function

$$U(c;\lambda) = \mathrm{E}\left[\int_0^T e^{-\rho t} u(c(t), \lambda(t))\, dt\right],$$

where u is the function in Equation (7) and λ is a (yet to be determined) weighting process. As in the unrestricted case, since consuming the aggregate endowment must be optimal for the representative agent, the marginal rate of substitution process

$$\xi(t) = e^{-\rho t} \frac{u_c(\delta(t), \lambda(t))}{u_c(\delta(0), \lambda(0))}$$

must identify the equilibrium state price density for the economy. Moreover, the optimal consumption allocation at time t must solve the representative agent problem in Equation (7) when $c = \delta(t)$ and $\lambda = \lambda(t)$, that is,

$$c_1^*(t) = f_1(u_c(\delta(t), \lambda(t))) \tag{18}$$

and

$$c_2^*(t) = f_2(u_c(\delta(t), \lambda(t))/\lambda(t)). \tag{19}$$

This shows that

$$\lambda(t) = \frac{u_1'(c_1^*(t))}{u_2'(c_2^*(t))}. \tag{20}$$

Now consider the optimal consumption problem for agent 1. Since this agent is facing a dynamically complete market, the optimality of $c_1^*(t)$ is equivalent to the marginal utility process $e^{-\rho t} u'(c_1^*(t))$ being proportional to the equilibrium state price density, that is,

$$e^{-\rho t} u_1'(c_1^*(t)) = \psi_1 \xi(t) \tag{21}$$

for some $\psi_1 > 0$. On the other hand, since agent 2 is facing a dynamically incomplete market, this proportionality will in general fail. In fact, since agent 2 has logarithmic preferences and can only invest in the bond, it follows from the results of He and Pearson (1991) and Karatzas et al. (1991) that the optimality of c_2^* is equivalent to

$$e^{-\rho t} u_2'(c_2^*(t)) = \psi_2 B(t)^{-1} \tag{22}$$

for some $\psi_2 > 0$. Equations (21) and (22) formalize the fact that, as a result of their different investment opportunity sets, the agents face different state-price densities. Applying Itô's lemma to Equation (20) and using Equations (21), (22), and (8) shows that

$$d\lambda(t) = -\lambda(t)\kappa(t)\,dw(t) = \frac{u_1''[f_1(u_c(\delta(t),\lambda(t)))]}{u_c(\delta(t),\lambda(t))}\lambda(t)\sigma_\delta(t)\,dw(t). \tag{23}$$

The following theorem formalizes the above discussion.

Theorem 1. *Suppose that there exists a strictly positive solution λ to the stochastic differential equation [Equation (23)] with $\lambda(0)$ being the unique positive solution of*

$$\beta = \frac{1-e^{-\rho T}}{\rho}\,\frac{\lambda(0)}{u_c(\delta(0),\lambda(0))} \tag{24}$$

and that

$$\mathrm{E}\left[\int_0^T e^{-\rho t}u_c(\delta(t),\lambda(t))\delta(t)\,dt\right] < \infty.$$

Then there exists an equilibrium for the economy $\mathcal{E}$, given by

$$r(t) = -\frac{\mathcal{D}(e^{-\rho t}u_c(\delta(t),\lambda(t)))}{e^{-\rho t}u_c(\delta(t),\lambda(t))}, \tag{25}$$

$$\begin{aligned} S(t) \;=\; & \mathrm{E}\left[\int_t^T e^{-\rho(s-t)}\frac{u_c(\delta(s),\lambda(s))}{u_c(\delta(t),\lambda(t))}f_1(u_c(\delta(s),\lambda(s)))\,ds\;\Big|\;\mathcal{F}_t\right] \\ & + \frac{1-e^{-\rho(T-t)}}{\rho}\,\frac{\lambda(t)}{u_c(\delta(t),\lambda(t))} \end{aligned} \tag{26}$$

and the consumption policies in Equations (18) and (19), where $\mathcal{D}(e^{-\rho t}u_c(\delta(t),\lambda(t)))$ denotes the drift of the process $e^{-\rho t}u_c(\delta(t),\lambda(t))$. The optimal trading strategies are given by

$$\alpha_1^*(t) = -\beta\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}B(t),$$

$\theta_1^*(t) = S(t)$, $\alpha_2^*(t) = -\alpha_1^*(t)$ and $\theta_2^*(t) = 0$.[5]

Equations (23) and (24) completely identify the weighting process λ, and hence the equilibrium. The existence of a solution to the stochastic differential equation [Equation (23)] can be easily guaranteed under appropriate conditions on the utility function u_1 and

[5] The model seems to have the counterfactual implication that stockholders (agent 1) are always net borrowers. On the other hand, as in Saito (1996), it would be possible to consider the riskless asset as a short-term liability of the firms which are held by agent 1, rather than a direct liability of the investors. In this case, depending on the amount of debt issued by the firm, agent 1 could end up with a long bond position. We thank Kerry Back for pointing this to us.

on the coefficients $(\mu_\delta, \sigma_\delta)$ of the dividend process. We will give an explicit existence proof for a specific example in the next section.

Remark 3. *The proof of Theorem 1 also shows that the bond price is given in closed form by*

$$B(t) = e^{\rho t} \frac{\lambda(t) u_c(\delta(0), \lambda(0))}{\lambda(0) u_c(\delta(t), \lambda(t))}.$$

Remark 4. *Equation (23) implies that the weighting process λ is a nonnegative local martingale, and hence a supermartingale. This is intuitive, since it implies that the weight of restricted agents in the economy is expected to decline over time (i.e., market participation is expected to increase) as a result of the larger investment opportunity set available to unrestricted agents. If λ is in fact a martingale, then Equation (26) reduces to the familiar stock price representation*

$$S(t) = \mathrm{E}\left[\int_t^T e^{-\rho(s-t)} \frac{u_c(\delta(s), \lambda(s))}{u_c(\delta(t), \lambda(t))} \delta(s)\, ds \,\Big|\, \mathcal{F}_t\right].$$

If the exogenously given dividend process δ is Markovian, then the equilibrium interest rate, stock price, and consumption policies in the unrestricted economy can be expressed as deterministic functions of the concurrent aggregate consumption and time, that is, δ is the only state variable. This is in general not true in the restricted economy. However, all the relevant past history of the economy is captured by the one-dimensional process λ, which appears as a second state variable, and the joint evolution of (δ, λ) is again Markovian. The stochastic weight λ placed on the restricted investor by the representative agent acts as a proxy for stochastic shifts in the distribution of wealth between the two agents. This process is negatively correlated with aggregate consumption (and hence with the stock price): since the restricted agent is prevented from investing in the stock market, a positive shock to consumption shifts wealth from the restricted to the unrestricted agent and hence reduces the weight assigned to the former by the representative agent. In fact, since the representative agent construction in Equation (7) would force both agents' consumption to be positively correlated with aggregate consumption, the equilibrium process λ acts so as to exactly cancel this correlation for the restricted agent.

Equation (23) also reveals that if the unrestricted agent exhibits decreasing absolute risk aversion, then the instantaneous variance rate of the weighting process, $(A_1\sigma_\delta)^2$, is decreasing in aggregate consumption and increasing in the level of the weighting process. An implication is that the variability in the distribution of consumption and wealth between stockholders and nonstockholders increases during periods of expansion and decreases during periods of recession. Furthermore, the variance of the weighting process decreases as the unrestricted agent's risk aversion decreases. Intuitively, since a less risk-averse agent 1 would hold more of the stock in the benchmark economy, there is a smaller impact of restricting agent 2 from participating in the stock market, and this is reflected by the weighting process having a lower variance around its value in the benchmark economy. In particular, in the extreme case of the unrestricted agent being risk neutral the weighting process becomes constant, consequently collapsing the restricted economy to the benchmark one.

The next two corollaries characterize the stochastic processes followed in equilibrium by the interest rate, the market price of risk, and the optimal consumption choices.

Corollary 1. *The equilibrium interest rate and risk premium processes have the representations*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}A(t)P_1(t)\sigma_\delta(t)^2$$

and

$$\mu(t) - r(t) = A_1(t)\sigma_\delta(t)\sigma(t).$$

In particular, if the stock volatility process does not vanish, then the market price of risk process is given by

$$\kappa(t) = A_1(t)\sigma_\delta(t).$$

Corollary 2. *The optimal consumption policies c_1^* and c_2^* satisfy*

$$dc_i^*(t) = \mu_{c_i^*}(t)\,dt + \sigma_{c_i^*}(t)\,dw(t),$$

where

$$\mu_{c_1^*}(t) = \frac{A(t)}{A_1(t)}\mu_\delta(t) - \frac{1}{2}\frac{A(t)}{A_1(t)}P_1(t)\sigma_\delta(t)^2 + \frac{1}{2}P_1(t)\sigma_\delta(t)^2,$$

$$\mu_{c_2^*}(t) = \frac{A(t)}{A_2(t)}\mu_\delta(t) - \frac{1}{2}\frac{A(t)}{A_2(t)}P_1(t)\sigma_\delta(t)^2,$$

$$\sigma_{c_1^*}(t) = \sigma_\delta(t)$$

and

$$\sigma_{c_2^*} \equiv 0.$$

Corollary 1 shows that, as in the unrestricted economy, the interest rate in the restricted economy is positively related to the impatience parameter and to the growth in aggregate consumption (with a sensitivity to the latter given by the representative risk aversion in the economy). Likewise, as long as agent 1 has positive prudence, the interest rate is negatively related to the aggregate consumption volatility, due to the precautionary saving motive (whose intensity is captured by the prudence coefficient) of agents who face uncertain future consumption.[6] Since Corollary 2 reveals that the unconstrained agent is now the only agent to face a locally risky consumption plan, the interest rate in the restricted economy responds directly only to the unrestricted agent's prudence, rather than to the representative agent's. Moreover, in the restricted economy the market price of risk κ only directly depends on the risk aversion of the unrestricted agent. The risk aversion and prudence of the restricted agent have only an indirect effect on the interest rate and the market price of risk through their effect on the equilibrium consumption allocation.

Corollary 2 shows that, as a consequence of restricting the logarithmic (myopic) agent to only invest in the bond, his consumption becomes locally riskless, and hence does not covary with aggregate consumption. The unrestricted agent must then absorb all the aggregate consumption risk into his consumption stream. Consequently the restricted agent's consumption volatility is unambiguously decreased in comparison to the unrestricted economy, while the unrestricted agent's is unambiguously increased.[7] As in the unrestricted economy, each agent's mean consumption growth in the restricted economy is positively related to the mean aggregate consumption growth, with a sensitivity inversely related to each agent's risk aversion. Moreover, each agent's consumption growth is negatively related, via the associated interest rate reduction, to the aggregate amount of precautionary savings, which in the restricted economy are due only to agent 1 (since he is now absorbing the aggregate consumption risk): this is captured by the second term in both agents' mean consumption growths. Finally, agent 1's mean consumption growth also reflects the positive direct effect of his amount of precautionary savings; no such term appears for the restricted agent, who has no precautionary savings motive.

Figures 1 and 2 plot the equilibrium interest rate and market price of risk implied by the model as a function of the nonstockholders'

[6] The notion of precautionary saving refers to the additional demand for saving due to uncertainty in future income (or, more generally, future consumption): see, for example, Leland (1968). The relationship between precautionary saving and the prudence coefficient was originally pointed out by Kimball (1990).

[7] This follows from the fact that $0 < \frac{1}{A(t)} = \frac{1}{A_1(t)} + \frac{1}{A_2(t)}$, so that $0 < \frac{A(t)}{A_i(t)}\sigma_\delta(t) < \sigma_\delta(t)$.

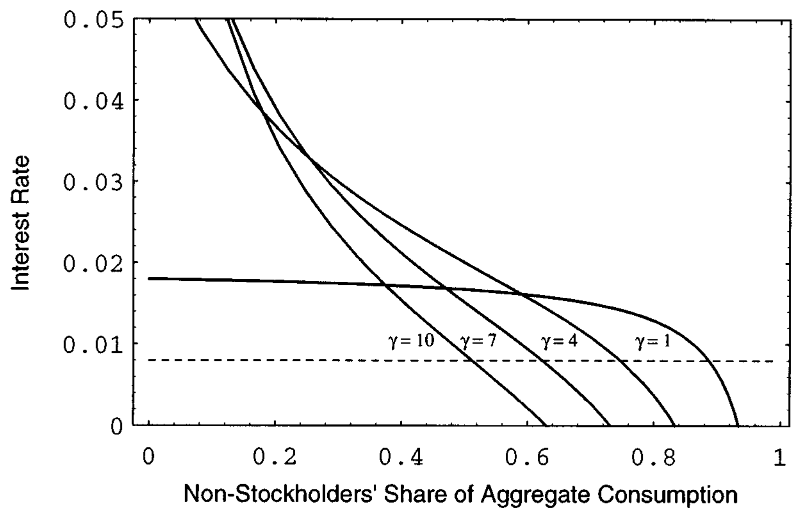

Figure 1
Behavior of the equilibrium interest rate
The graph plots the equilibrium interest rate implied by the model as a function of the nonstockholders' share of aggregate consumption, assuming that stockholders display CRRA preferences with a relative risk aversion coefficient γ equal to 1, 4, 7, and 10. The parameters of the aggregate consumption process are chosen to match the estimates reported by Mehra and Prescott (1985). The dotted line corresponds to Mehra and Prescott's estimate of the mean real interest rate.

share of aggregate consumption x (which identifies the weight λ), assuming the stockholders display CRRA preferences with a relative risk aversion coefficient γ equal to 1, 4, 7, and 10.[8] We choose the parameters of the aggregate consumption process δ so as to match those estimated in Mehra and Prescott (1985) for per capita real consumption of nondurables and services in the years 1889 to 1978, that is, we set $\mu_\delta(t)/\delta(t) = 0.0183$ and $\sigma_\delta(t)/\delta(t) = 0.0357$, while we take $\rho = 0.001$. Mehra and Prescott also estimated the mean real interest rate to be 0.008 and the mean and the standard deviation of the real rate of return on the Standard & Poor's composite stock price index to be, 0.0698 and 0.1654, respectively. These estimates imply a market price of risk of about 0.37.

The first result that emerges from Figures 1 and 2 is a confirmation of the well-documented pricing puzzle: the standard complete market model (which corresponds in our setting to $x = 0$) implies an interest

[8] We normalize aggregate consumption δ to 1, so that $\lambda = u_1'(1-x)/u_2'(x)$. This is without loss of generality with isoelastic preferences and an aggregate dividend following a geometric Brownian motion, as the expressions for the interest rate and the market price of risk are in this case homogeneous of degree zero in δ keeping the consumption share x constant.

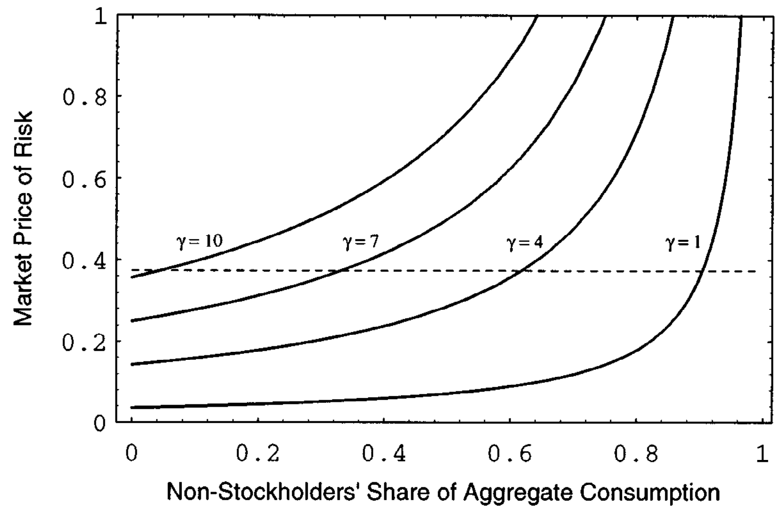

Figure 2
Behavior of the equilibrium market price of risk
The graph plots the equilibrium market price of risk implied by the model as a function of the nonstockholders' share of aggregate consumption, assuming that stockholders display CRRA preferences with a relative risk aversion coefficient γ equal to 1, 4, 7, and 10. The parameters of the aggregate consumption process are chosen to match the estimates reported by Mehra and Prescott (1985). The dotted line corresponds to Mehra and Prescott's estimate of the market price of risk.

rate too high and a market price of risk too low (at least for "reasonable" values of aggregate risk tolerance). In particular, the historical estimate of the market risk premium would imply a representative agent's risk aversion coefficient of 10.5, which in turn corresponds to a real interest rate of 11.6%. On the other hand, assuming a nonstockholders' share of aggregate consumption of 0.68 (which is the Mankiw and Zeldes' estimate of the nonstockholders' share of food expenditure), our model can match the historical market price of risk at $\gamma = 3.3$, which corresponds to a real interest rate of 1.3% and a representative agent's relative risk aversion of 1.3, well within the "reasonable" range.

4. An Example

To further explore the impact of restricted market participation on equilibrium, we now specialize our economy to the case in which both agents have logarithmic preferences (i.e., $u_1(c) = u_2(c) = \log c$) and the aggregate consumption process follows a geometric Brownian motion, that is,

$$\delta(t) = \delta(0) + \int_0^t \delta(s)\bar{\mu}_\delta \, ds + \int_0^t \delta(s)\bar{\sigma}_\delta \, dw(s)$$

for some constants $\bar{\mu}_\delta \in \mathbb{R}$ and $\bar{\sigma}_\delta > 0$. We refer to this economy as $\mathcal{E}'$.

The representative agent's utility function is given by

$$u(c, \lambda) = \max_{c_1 + c_2 = c} u_1(c_1) + \lambda u_2(c_2) = \log\left(\frac{c}{1+\lambda}\right) + \lambda \log\left(\frac{\lambda c}{1+\lambda}\right),$$

so that $A(t) = \delta(t)^{-1}$ and $P(t) = 2\delta(t)^{-1}$.

In the unrestricted case, the equilibrium interest rate, stock price, and consumption allocations are then

$$r(t) = \rho + \bar{\mu}_\delta - \bar{\sigma}_\delta^2,$$

$$S(t) = \frac{1 - e^{-\rho(T-t)}}{\rho} \delta(t),$$

$$c_1^*(t) = \frac{\delta(t)}{1+\lambda} = \frac{\delta(0)(1 - e^{-\rho T}) - \rho\beta}{\delta(0)(1 - e^{-\rho T})} \delta(t)$$

and

$$c_2^*(t) = \frac{\lambda\delta(t)}{1+\lambda} = \frac{\rho\beta}{\delta(0)(1 - e^{-\rho T})} \delta(t),$$

where

$$\lambda = \frac{\rho\beta}{\delta(0)(1 - e^{-\rho T}) - \rho\beta}.$$

This implies $\mu(t) = \rho + \bar{\mu}_\delta$, $\sigma(t) = \bar{\sigma}_\delta$, and

$$dc_i^*(t) = c_i^*(t)\bar{\mu}_{c_i^*} \, dt + c_i^*(t)\bar{\sigma}_{c_i^*} \, dw(t),$$

where $\bar{\mu}_{c_i^*} = \bar{\mu}_\delta$ and $\bar{\sigma}_{c_i^*} = \bar{\sigma}_\delta$. Furthermore, the optimal trading strategies are $\alpha_i^*(t) = \alpha_2^*(t) = 0$, $\theta_1^*(t) = S(t)/(1+\lambda)$, and $\theta_2^*(t) = \lambda S(t)/(1+\lambda)$.

We now turn to the restricted economy. Under our current assumptions, the stochastic differential equation [Equation (23)] becomes

$$d\lambda(t) = -(\lambda(t) + \lambda(t)^2)\bar{\sigma}_\delta \, dw(t), \tag{27}$$

with initial condition

$$\lambda(0) = \frac{\rho\beta}{\delta(0)(1 - e^{-\rho T}) - \rho\beta}. \tag{28}$$

Lemma 1. *There exists a unique strictly positive process λ solving the stochastic differential equation [Equation (27)] with the initial condition of Equation (28).*

As a consequence of the above lemma and Theorem 1, it is immediate to show the existence of an equilibrium for $\mathcal{E}'$.

Theorem 2. *There exists an equilibrium for the economy $\mathcal{E}'$, given by*

$$r(t) = \rho + \bar{\mu}_\delta - (1 + \lambda(t))\bar{\sigma}_\delta^2, \tag{29}$$

$$S(t) = \frac{1 - e^{-\rho(T-t)}}{\rho}\,\delta(t) \tag{30}$$

and the consumption policies $c_1^*(t) = \delta(t)/(1 + \lambda(t))$ and $c_2^*(t) = \lambda(t)\delta(t)/(1 + \lambda(t))$, where λ solves the stochastic differential equation [Equation (27)] with the initial condition of Equation (28). The optimal trading strategies are as in Theorem 1.

Remark 5. *It follows from Remark 3 that the bond price process is given in closed form by*

$$B(t) = e^{\rho t}\,\frac{1 + \lambda(0)}{\delta(0)\lambda(0)}\,\frac{\delta(t)\lambda(t)}{1 + \lambda(t)}.$$

Corollary 3. *The equilibrium interest rate, given by Equation (29), satisfies*

$$dr(t) = (\lambda(t) + \lambda(t)^2)\bar{\sigma}_\delta^2\, dw(t).$$

Furthermore, the interest rate in the restricted economy is lower than in the unrestricted economy, and it decreases as initial market participation decreases (as captured by an increasing β).

The equilibrium market price of risk, given by

$$\kappa(t) = (1 + \lambda(t))\bar{\sigma}_\delta,$$

satisfies

$$d\kappa(t) = -(\lambda(t) + \lambda(t)^2)\bar{\sigma}_\delta^2\, dw(t).$$

The market price of risk in the restricted economy is higher than in the unrestricted economy, and it increases as initial market participation decreases.

Corollary 4. *The optimal consumption policies satisfy*

$$dc_i^*(t) = c_i^*(t)\bar{\mu}_{c_i^*}(t)\, dt + c_i^*(t)\bar{\sigma}_{c_i^*}\, dw(t),$$

where

$$\bar{\mu}_{c_1^*}(t) = \bar{\mu}_\delta + (\lambda(t) + \lambda(t)^2)\bar{\sigma}_\delta^2,$$

$$\bar{\mu}_{c_2^*}(t) = \bar{\mu}_\delta - (\lambda(t) + \lambda(t)^2)\bar{\sigma}_\delta^2,$$

$$\bar{\sigma}_{c_1^*} = (1 + \lambda(t))\bar{\sigma}_\delta$$

and

$$\bar{\sigma}_{c_2^*} = 0.$$

Consequently the unrestricted agent's expected consumption growth rate is higher and his consumption volatility is unchanged relative to the benchmark economy, while the restricted agent's expected consumption growth rate and consumption volatility are lower than in the benchmark economy.

In both the unrestricted and restricted economy with logarithmic agents, the weighting process λ equals the ratio of the agents' consumption or wealth processes (i.e., $\lambda = c_2^*/c_1^* = W_2/W_1$). In the restricted economy this ratio fluctuates randomly. The resulting changes in the distribution of wealth between the two agents affect both the equilibrium interest rate and the equilibrium market price of risk (which become stochastic), but not the equilibrium stock expected return and volatility, which remain as in the unrestricted economy. A redistribution of wealth in favor of the restricted agent (i.e., an increase of λ) decreases the interest rate, increases the market risk premium and makes both the interest rate and the market risk premium more volatile.

Since the weighting process λ is negatively correlated with aggregate consumption, the interest rate in the restricted economy is procyclical, while the risk premium is anticyclical. This should be intuitive. As aggregate consumption increases, the stock price increases and, since the unrestricted agent is a net borrower, the fraction of his wealth invested in the stock market tends to decrease. If the price coefficients r, μ, and σ were to remain constant, he would then choose to rebalance his portfolio to purchase more stock by borrowing more.[9] In other words, he would behave as a "trend-chaser." On the other hand, since in equilibrium the unrestricted agent must always hold the aggregate supply of the stock, the price coefficients must adjust so as to make the stock less favorable relative to the bond. Since the equilibrium stock price coefficients μ and σ are constant, the only way to achieve this is for the interest rate to increase (and hence for the risk premium to decrease).

[9] This is because the optimal stock holding for a logarithmic investor is given by $\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)^2} W(t)$.

Corollary 3 also reveals that the interest rate in the restricted economy is unambiguously lower, and the market price of risk unambiguously higher, than in the unrestricted economy.

To provide intuition for this result, note that in equilibrium in the benchmark economy there is no borrowing or lending between the two logarithmic agents. In the restricted economy, however, the restricted agent can only invest in the bond, and thus must lend to the unrestricted agent. For an equilibrium to be achieved, the unrestricted agent must therefore be induced to become a net borrower. Since the equilibrium stock price is unaffected by the market participation restriction, this implies that the interest rate must decrease and the market price of risk must increase. Furthermore, this effect must be more pronounced the larger the initial share of restricted agents in the economy (as measured by β).

Finally, Corollary 4 reveals that, under the assumptions of this section, the unrestricted agent's expected percentage consumption growth is unambiguously higher than in the benchmark economy, while the restricted agent's is lower.

Restricted market participation also has unambiguous effects on the agents' welfare.

Proposition 1. *Let c_i^u and c_i^r denote, respectively, the optimal consumption choices of agent i in the unrestricted and restricted economies. Then*

$$U_1(c_1^r) > U_1(c_1^u)$$

and

$$U_2(c_2^r) < U_2(c_2^u).$$

Proposition 1 reveals that the restricted agent is made worse off by restricting his participation in the stock market, while the unrestricted agent is made better off, even though he is absorbing more risk into his consumption than in the benchmark economy. If prices in the restricted economy were the same as in the benchmark economy, preventing the second agent from trading in the stock market would clearly make him worse off, since otherwise he would have chosen not to trade the stock in the benchmark economy. Moreover, since the second agent must be a net lender in the restricted economy and the interest rate must decrease relative to the benchmark economy, he is also negatively affected by the change in the terms of trade. On the other hand, the lower interest rate clearly benefits the unrestricted agent, since he was neither a borrower nor a lender in the benchmark economy and becomes a net borrower in the restricted economy.

5. Conclusion

This article studies a continuous-time pure-exchange economy populated by stockholders and nonstockholders. An equilibrium is explicitly constructed using a representative agent utility function assigning possibly stochastic weights to the agents in the economy. We provide a full characterization of equilibrium prices and optimal consumption/investment policies and investigate the equilibrium implications of restricted market participation. The model's conclusions are shown to be consistent with the empirical regularities exhibited by consumption and financial data: in particular, given a nonstockholders' share of aggregate consumption of 68% [as reported by Mankiw and Zeldes (1991)], the model can match the historical market price of risk for a level of aggregate relative risk aversion of about 1.3.

Appendix

Lemma 2. *The identities*

$$f_1(u_c(c,\lambda)) + f_2(u_c(c,\lambda)/\lambda) = c, \tag{31}$$

$$\frac{1}{u_{cc}(c,\lambda)} = \frac{1}{u_1''(f_1(u_c(c,\lambda)))} + \frac{1}{\lambda u_2''(f_1(u_c(c,\lambda)/\lambda))} \tag{32}$$

and

$$u_1''(f_1(u_c(c,\lambda))) = \frac{u_c(c,\lambda)u_{cc}(c,\lambda)}{u_c(c,\lambda) - \lambda u_{c\lambda}(c,\lambda)} \tag{33}$$

hold for $(c,\lambda) \in (0,\infty) \times (0,\infty)$.

Proof. For the first identity see, for example, Karatzas, Lehoczky, and Shreve (1990, Lemma 10.1). The second identity follows from differentiating Equation (31) with respect to c and using the fact that $f_i'(x) = 1/u_i''(f_i(x))$. Finally, differentiating Equation (31) with respect to λ gives

$$\begin{aligned} 0 &= f_1'(u_c(c,\lambda))u_{c\lambda}(c,\lambda) + f_2'(u_c(c,\lambda)/\lambda)\frac{\lambda u_{c\lambda}(c,\lambda) - u_c(c,\lambda)}{\lambda^2} \\ &= \frac{u_{c\lambda}(c,\lambda)}{u_1''(f_1(u_c(c,\lambda)))} + \frac{1}{\lambda u_2''(f_2(u_c(c,\lambda)/\lambda))}\frac{\lambda u_{c\lambda}(c,\lambda) - u_c(c,\lambda)}{\lambda} \end{aligned}$$

$$
\begin{aligned}
&= \frac{u_{c\lambda}(c,\lambda)}{u_1''(f_1(u_c(c,\lambda)))} \\
&\quad + \left(\frac{1}{u_{cc}(c,\lambda)} - \frac{1}{u_1''(f_1(u_c(c,\lambda)))}\right)\frac{\lambda u_{c\lambda}(c,\lambda) - u_c(c,\lambda)}{\lambda} \\
&= \frac{\lambda u_{c\lambda}(c,\lambda) - u_c(c,\lambda)}{\lambda u_{cc}(c,\lambda)} + \frac{u_c(c,\lambda)}{\lambda u_1''(f_1(u_c(c,\lambda)))},
\end{aligned}
$$

where the third equality follows from Equation (32). Rearranging this last expression gives Equation (33). ■

Proof of Theorem 1. We proceed in two steps: first we show that the given consumption policies c_i^* are financed by the specified trading strategies (α_i^*, θ_i^*), and then that c_i^* is optimal for agent i. Since $\alpha_1^* + \alpha_2^* = 0$, $\theta_1^* + \theta_2^* = S$, and Equation (31) implies that $c_1^* + c_2^* = \delta$, this verifies that the given asset prices, consumption policies, and trading strategies constitute an equilibrium for the economy $\mathcal{E}$.

To simplify the notation we will write $u_c(t)$ for $u_c(\delta(t), \lambda(t))$ and $u_i'(t)$ for $u_i'(c_i^*(t))$, with a similar convention for the higher-order derivatives. We remark that it follows immediately from Equations (18) and (19) that $u_1'(t) = u_c(t)$ and $u_2'(t) = u_c(t)/\lambda(t)$.

Step 1. By Itô's lemma, Equations (1), (23), (25), and (33), we have

$$
\begin{aligned}
d\left(e^{-\rho t} u_c(t)\right) &= -e^{-\rho t} u_c(t) r(t)\, dt \\
&\quad + e^{-\rho t}\left(u_{cc}(t) + \lambda(t)\frac{u_1''(t)}{u_c(t)} u_{c\lambda}(t)\right)\sigma_\delta(t)\, dw(t) \\
&= -e^{-\rho t} u_c(t) r(t)\, dt + e^{-\rho t} u_1''(t)\sigma_\delta(t)\, dw(t), \qquad (34)
\end{aligned}
$$

and

$$
d\left(\frac{e^{-\rho t} u_c(t)}{\lambda(t)}\right) = -\frac{e^{-\rho t} u_c(t)}{\lambda(t)} r(t)\, dt. \qquad (35)
$$

The last equation implies

$$
B(t) = e^{\rho t}\frac{\lambda(t) u_c(0)}{\lambda(0) u_c(t)}, \qquad (36)
$$

and hence we have from Equations (19) and (24)

$$
c_2^*(t) = \frac{\lambda(t)}{u_c(t)} = \frac{\lambda(0)}{u_c(0)} e^{-\rho t} B(t) = \beta \frac{\rho e^{-\rho t}}{1 - e^{-\rho T}} B(t).
$$

It is now easy to verify that c_2^* is financed by the given trading strategy (α_2^*, θ_2^*), since we have

$$\begin{aligned} dW_2(t) &= d\left(\beta\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}B(t)\right) \\ &= \beta\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}B(t)r(t)\,dt - \beta\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}B(t)\,dt \\ &= \alpha_2^*(t)r(t)\,dt - c_2^*(t)\,dt. \end{aligned}$$

Moreover, $W_2 \geq 0$ and c_2^* is budget feasible for agent 2, since $W_2(0) = \alpha_2^*(0) = \beta$. Also, since $c_1^*(t)+c_2^*(t) = \delta(t)$ and $W_1(t)+W_2(t) = S(t)$, it is immediately verified that c_1^* is financed by the given trading strategy and is budget feasible for agent 1. Finally, it is easily verified that $W_1(t) \geq 0$ for all $t \in [0, T]$, as

$$\begin{aligned} W_1(t) &= S(t) - \beta\frac{e^{-\rho t}-e^{-\rho T}}{1-e^{-\rho T}}\,B(t) \\ &= S(t) - \frac{1-e^{-\rho(T-t)}}{\rho}\,\frac{\lambda(t)}{u_c(t)} \\ &= \mathrm{E}\left[\int_t^T e^{-\rho(s-t)}\frac{u_c(s)}{u_c(t)}f_1(u_c(s))\,ds \,\Big|\, \mathcal{F}_t\right], \end{aligned}$$

because of Equations (24), (26), and (36).

Step 2. Consider the process

$$\begin{aligned} M(t) &= e^{-\rho t}u_c(t)S(t) + \int_0^t e^{-\rho s}u_c(s)f_1(u_c(s))\,ds - \frac{e^{-\rho t}-e^{-\rho T}}{\rho}\,\lambda(t) \\ &= \mathrm{E}\left[\int_0^T e^{-\rho s}u_c(s)f_1(u_c(s))\,ds \,\Big|\, \mathcal{F}_t\right]. \end{aligned}$$

Since M is a martingale, we must have

$$\begin{aligned} 0 &= \mathcal{D}M(t) \\ &= e^{-\rho t}u_c(t)S(t)\left(\mu(t) - r(t) + \frac{u_1''(t)}{u_c(t)}\sigma_\delta(t)\sigma(t)\right) \\ &\quad - e^{-\rho t}u_c(t)\left(\delta(t) - f_1(u_c(t)) - \frac{\lambda(t)}{u_c(t)}\right) \\ &= e^{-\rho t}u_c(t)S(t)\left(\mu(t) - r(t) + \frac{u_1''(t)}{u_c(t)}\sigma_\delta(t)\sigma(t)\right), \end{aligned}$$

where the second equality follows from Itô's lemma, using Equations (3), (23), and (34), while the last equality follows from Equation (31) and the fact that $f_2(x) = 1/x$. Therefore

$$\mu(t) - r(t) = -\frac{u_1''(t)}{u_c(t)}\sigma_\delta(t)\sigma(t). \tag{37}$$

Now, let $c \in \mathcal{C}$ be an arbitrary consumption process financed by a trading strategy $(\alpha, \theta) \in \Theta$ with $\alpha(0) + \theta(0) = S(0) - \beta$ and let $W = \alpha + \theta$ denote the corresponding wealth process. By Itô's lemma and Equations (4), (34), and (37)

$$\begin{aligned}
d\left(\frac{e^{-\rho t}u_c(t)}{u_c(0)}W(t)\right) &= \frac{e^{-\rho t}u_c(t)}{u_c(0)}\theta(t)\left(\mu(t) - r(t) + \frac{u_1''(t)}{u_c(t)}\sigma_\delta(t)\sigma(t)\right)dt \\
&\quad - \frac{e^{-\rho t}u_c(t)}{u_c(0)}c(t)\,dt \\
&\quad + \frac{e^{-\rho t}u_c(t)}{u_c(0)}\left(\theta(t)\sigma(t) + \frac{u_1''(t)}{u_c(t)}W(t)\sigma_\delta(t)\right)dw(t) \\
&= - \frac{e^{-\rho t}u_c(t)}{u_c(0)}c(t)\,dt \\
&\quad + \frac{e^{-\rho t}u_c(t)}{u_c(0)}\left(\theta(t)\sigma(t) + \frac{u_1''(t)}{u_c(t)}W(t)\sigma_\delta(t)\right)dw(t).
\end{aligned}$$

This shows that the process

$$\begin{aligned}
&\frac{e^{-\rho t}u_c(t)}{u_c(0)}W(t) + \int_0^t \frac{e^{-\rho s}u_c(s)}{u_c(0)}c(s)\,ds \\
&\quad = W(0) + \int_0^t \frac{e^{-\rho s}u_c(s)}{u_c(0)}\left(\theta(s)\sigma(s) + \frac{u_1''(s)}{u_c(s)}W(s)\sigma_\delta(s)\right)dw(s)
\end{aligned}$$

is a nonnegative local martingale, and hence a supermartingale. Therefore

$$\begin{aligned}
S(0) - \beta \;=\; W(0) &\geq \mathrm{E}\left[\frac{e^{-\rho T}u_c(T)}{u_c(0)}W(T) + \int_0^T \frac{e^{-\rho t}u_c(t)}{u_c(0)}c(t)\,dt\right] \\
&\geq \mathrm{E}\left[\int_0^T \frac{e^{-\rho t}u_c(t)}{u_c(0)}c(t)\,dt\right].
\end{aligned}$$

By the concavity of u_1, we then have

$$\begin{aligned}
\frac{U_1(c) - U_1(c_1^*)}{u_c(0)} &\leq \mathrm{E}\left[\int_0^T \frac{e^{-\rho t}u_1'(t)}{u_c(0)}(c(t) - c_1^*(t))\,dt\right] \\
&= \mathrm{E}\left[\int_0^T \frac{e^{-\rho t}u_c(t)}{u_c(0)}(c(t) - c_1^*(t))\,dt\right]
\end{aligned}$$

$$\begin{aligned}&\leq\; S(0)-\beta-\mathrm{E}\left[\int_0^T \frac{e^{-\rho t}u_c(t)}{u_c(0)} f_1(u_c(t))\,dt\right]\\&=\;\frac{1-e^{-\rho T}}{\rho}\,\frac{\lambda(0)}{u_c(0)}-\beta\;=\;0,\end{aligned}$$

where the second equality follows from Equation (26), while the third equality follows from Equation (24). This shows the optimality of c_1^* for agent 1.

Similarly, in order to show the optimality of c_2^* for agent 2, let $c\in\mathcal{C}$ be an arbitrary consumption process financed by a trading strategy $(\alpha,\theta)\in\Theta$ with $\alpha(0)+\theta(0)=\beta$ and $\theta\equiv 0$, and let W be the associated wealth process. By Itô's lemma and Equations (4) and (35)

$$d\left(\frac{\lambda(0)e^{-\rho t}u_c(t)}{\lambda(t)u_c(0)}W(t)\right)=-\frac{\lambda(0)e^{-\rho t}u_c(t)}{\lambda(t)u_c(0)}c(t)\,dt,$$

so that

$$\begin{aligned}\beta\;&=\;W(0)=\frac{\lambda(0)}{u_c(0)}\,\mathrm{E}\left[\frac{e^{-\rho T}u_c(T)}{\lambda(T)}W(T)+\int_0^T\frac{e^{-\rho t}u_c(t)}{\lambda(t)}c(t)\,dt\right]\\&\geq\;\frac{\lambda(0)}{u_c(0)}\,\mathrm{E}\left[\int_0^T\frac{e^{-\rho t}u_c(t)}{\lambda(t)}c(t)\,dt\right].\end{aligned}$$

By the concavity of u_2, we then have

$$\begin{aligned}\frac{U_2(c)-U_2(c_2^*)}{u_c(0)/\lambda(0)}\;&\leq\;\mathrm{E}\left[\int_0^T\frac{\lambda(0)e^{-\rho t}u_2'(t)}{u_c(0)}(c(t)-c_2^*(t))\,dt\right]\\&=\;\mathrm{E}\left[\int_0^T\frac{\lambda(0)e^{-\rho t}u_c(t)}{\lambda(t)u_c(0)}(c(t)-c_2^*(t))\,dt\right]\\&\leq\;\beta-\frac{\lambda(0)}{u_c(0)}\,\mathrm{E}\left[\int_0^T\frac{e^{-\rho t}u_c(t)}{\lambda(t)}\,\frac{\lambda(t)}{u_c(t)}\,dt\right]\\&=\;\beta-\frac{1-e^{-\rho T}}{\rho}\,\frac{\lambda(0)}{u_c(0)}\;=\;0,\end{aligned}$$

where the last equality again follows from Equation (24). This shows the optimality of c_2^* for agent 2. ■

Proof of Corollary 1. Since

$$f_1(u_c(t))+f_2(u_c(t)/\lambda(t))=\delta(t),$$

we have by Itô's lemma and Equations (1), (34), and (35)

$$\begin{aligned}\mu_\delta(t) &= \left(f_1'(u_c(t)) + \frac{1}{\lambda} f_2'(u_c(t)/\lambda(t))\right) u_c(t)(\rho - r(t)) \\ &\quad + \frac{1}{2} f_1''(u_c(t)) u_1''(t)^2 \sigma_\delta(t)^2 \\ &= \frac{u_c(t)}{u_{cc}(t)}(\rho - r(t)) - \frac{1}{2}\frac{u_1'''(t)}{u_1''(t)}\sigma_\delta(t)^2,\end{aligned}$$

where the last equality follows from Equation (32) and the fact that $f_i'(y) = 1/u_i''(f_i(y))$ and $f_i''(y) = -u_i'''(f_i(y))/u_i''(f_i(y))^3$. Rearranging establishes the interest rate formula. The expression for the risk premium comes from Equation (37). ■

Proof of Corollary 2. Applying Itô's lemma to Equations (18) and (19) and using Equations (34) and (35) gives

$$\begin{aligned}dc_1^*(t) &= \left(f_1'(u_c(t))u_c(t)(\rho - r(t)) + \frac{1}{2} f_1''(u_c(t)) u_1''(t)^2 \sigma_\delta(t)^2\right) dt \\ &\quad + f_1'(u_c(t)) u_1''(t) \sigma_\delta(t)\, dw(t) \\ &= \left(-\frac{u_1'(t)}{u_1''(t)} A(t)\mu_\delta(t) + \frac{1}{2}\frac{u_1'(t)}{u_1''(t)} A(t) P_1(t)\sigma_\delta(t)^2 \right. \\ &\quad \left. - \frac{1}{2}\frac{u_1'''(t)}{u_1''(t)}\sigma_\delta(t)^2\right) dt + \sigma_\delta(t)\, dw(t) \\ &= \left(\frac{A(t)}{A_1(t)}\mu_\delta(t) - \frac{1}{2}\frac{A(t)}{A_1(t)} P_1(t)\sigma_\delta(t)^2 + \frac{1}{2} P_1(t)\sigma_\delta(t)^2\right) dt \\ &\quad + \sigma_\delta(t)\, dw(t)\end{aligned}$$

and

$$\begin{aligned}dc_2^*(t) &= f_2'(u_c(t)/\lambda(t))\frac{u_c(t)}{\lambda(t)}(\rho - r(t))\, dt \\ &= \left(-\frac{u_2'(t)}{u_2''(t)} A(t)\mu_\delta(t) + \frac{1}{2}\frac{u_2'(t)}{u_2''(t)} A(t) P_1(t)\sigma_\delta(t)^2\right) dt \\ &= \left(\frac{A(t)}{A_2(t)}\mu_\delta(t) - \frac{1}{2}\frac{A(t)}{A_2(t)} P_1(t)\sigma_\delta(t)^2\right) dt.\end{aligned}$$ ■

Proof of Lemma 1. Writing the stochastic differential equation [Equation (27)] as

$$d\lambda(t) = \sigma_\lambda(\lambda(t))\, dw(t),$$

where $\sigma_\lambda(x) = -(x + x^2)$, and defining the set

$$I(\sigma_\lambda) = \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \left(\frac{1}{\sigma_\lambda(y)} \right)^2 dy = \infty,\ \forall \varepsilon > 0 \right\},$$

it can be verified that $I = \{\{-1\}, \{0\}\}$. Moreover, setting $f(x) = 3 + x^2$ and $h(x) = x$, we have

$$|\sigma_\lambda(x + y) - \sigma_\lambda(x)| \leq f(x) h(|y|) \qquad \text{for all } x \in \mathbb{R},\ y \in [-1, 1]$$

and the function $(f/\sigma_\lambda)^2$ is locally integrable on $I(\sigma_\lambda)^c$, that is, for all $x \in \mathbb{R} \backslash I(\sigma_\lambda)$ there exists an $\varepsilon > 0$ such that

$$\int_{x-\varepsilon}^{x+\varepsilon} \left(\frac{f(y)}{\sigma_\lambda(y)} \right)^2 dy < \infty.$$

The existence of a unique (strong) solution then follows from Corollary 5.5.10 in Karatzas and Shreve (1988).

To show that the solution is strictly positive, define the *scale function*

$$p(x) = x - 1$$

and the *speed measure*

$$m(dx) = \frac{2dx}{p'(x)\sigma_\lambda(x)^2} = \frac{2dx}{x^2(1 + x)^2},$$

as well as the function

$$v(x) = \int_1^x (p(x) - p(y))\, m(dy) = 2(1 + 2x) \log\left(\frac{1 + x}{2x} \right) + 3(x - 1).$$

Since $v(0) = v(\infty) = \infty$, it follows from Theorem 5.5.29 in Karatzas and Shreve (1988) that

$$P\left(\{\omega : 0 < \lambda(t, \omega) < \infty,\ \forall t \in [0, T]\}\right) = 1. \qquad \blacksquare$$

Proof of Theorem 2. By Lemma 1 there exists a strictly positive solution λ to the stochastic differential equation [Equation (27)] with the initial condition of Equation (28). Moreover, since λ is a strictly positive local martingale, and hence a supermartingale, we have

$$\begin{aligned} \mathrm{E}\left[\int_0^T e^{-\rho t} \frac{u_c(t)}{u_c(0)} \delta(t)\, dt \right] &= \frac{\delta(0)}{1 + \lambda(0)} \mathrm{E}\left[\int_0^T e^{-\rho t}(1 + \lambda(t))\, dt \right] \\ &\leq \delta(0) \frac{1 - e^{-\rho T}}{\rho} < \infty. \end{aligned}$$

It then follows from Theorem 1 that there exists an equilibrium for the economy $\mathcal{E}'$, given by

$$r(t) = -\frac{\mathcal{D}(e^{-\rho t} u_c(t))}{e^{-\rho t} u_c(t)} = \rho + \bar{\mu}_\delta - (1+\lambda(t))\bar{\sigma}_\delta^2,$$

$$\begin{aligned} S(t) \;&=\; \mathrm{E}\left[\int_t^T e^{-\rho(s-t)} \frac{u_c(s)}{u_c(t)} f_1(u_c(s))\, ds \,\middle|\, \mathcal{F}_t\right] \\ &\quad + \frac{1-e^{-\rho(T-t)}}{\rho} \frac{\lambda(t)}{u_c(t)} \\ &= \frac{1-e^{-\rho(T-t)}}{\rho} \frac{1+\lambda(t)}{u_c(t)} = \frac{1-e^{-\rho(T-t)}}{\rho} \delta(t), \end{aligned}$$

(where the second equality follows from the fact that $f_1(x) = 1/x$ and the last equality from the fact that $u_c(c, \lambda) = (1+\lambda)/\delta)$ and the consumption policies

$$c_1^*(t) = f_1(u_c(\delta(t), \lambda(t)) = \frac{\delta(t)}{1+\lambda(t)}$$

and

$$c_2^*(t) = f_2(u_c(\delta(t), \lambda(t))/\lambda(t)) = \frac{\lambda(t)\delta(t)}{1+\lambda(t)}.$$

Moreover, the optimal trading strategies are given by

$$\alpha_1^*(t) = -\beta \frac{e^{-\rho t} - e^{-\rho T}}{1-e^{-\rho T}} B(t),$$

$\theta_1^*(t) = S(t)$, $\alpha_2^*(t) = -\alpha_1^*(t)$, and $\theta_2^*(t) = 0$. ■

Proof of Corollary 3. Applying Itô's lemma to Equation (29) and using Equation (27) gives the interest rate dynamics. Equation (29), the fact that $\lambda(t)$ is increasing in the initial condition $\lambda(0)$ for all $t \in [0, T]$ [cf. Theorem V.39 in Protter (1990)] and Equation (28) yield the comparison of interest rates across economies. The expression for the market price of risk process follows from the fact that Equation (30) implies $\mu(t) = \rho + \bar{\mu}_\delta$ and $\sigma(t) = \bar{\sigma}_\delta$. Its dynamics and the comparison across economies are obtained as for the interest rate. ■

Proof of Corollary 4. Obvious using Itô's lemma and the expressions for the optimal consumption policies. ■

Proof of Proposition 1. For the unrestricted agent, we have

$$\begin{aligned}
U_1(c_1^r) - U_1(c_1^u) &= \mathrm{E}\left[\int_0^T e^{-\rho t}\log\left(\frac{\delta(t)}{1+\lambda(t)}\right)dt\right] \\
&\quad - \mathrm{E}\left[\int_0^T e^{-\rho t}\log\left(\frac{\delta(t)}{1+\lambda(0)}\right)dt\right] \\
&= \frac{1-e^{-\rho T}}{\rho}\log(1+\lambda(0)) \\
&\quad - \mathrm{E}\left[\int_0^T e^{-\rho t}\log(1+\lambda(t))\,dt\right] \\
&> \frac{1-e^{-\rho T}}{\rho}\log(1+\lambda(0)) \\
&\quad - \int_0^T e^{-\rho t}\log(1+\lambda(0))\,dt = 0,
\end{aligned}$$

where the inequality follows from Jensen's inequality and the fact that λ is a supermartingale. Similarly, since the function $g(\lambda) = \log(\lambda/(1+\lambda))$ is increasing and concave for $\lambda > 0$ and λ is a supermartingale, we have for the restricted agent

$$\begin{aligned}
U_2(c_2^r) - U_2(c_2^u) &= \mathrm{E}\left[\int_0^T e^{-\rho t}\log\left(\frac{\lambda(t)\delta(t)}{1+\lambda(t)}\right)dt\right] \\
&\quad - \mathrm{E}\left[\int_0^T e^{-\rho t}\log\left(\frac{\lambda(0)\delta(t)}{1+\lambda(0)}\right)dt\right] \\
&= \mathrm{E}\left[\int_0^T e^{-\rho t}\log\left(\frac{\lambda(t)}{1+\lambda(t)}\right)dt\right] \\
&\quad - \frac{1-e^{-\rho T}}{\rho}\log\left(\frac{\lambda(0)}{1+\lambda(0)}\right) \\
&< \int_0^T e^{-\rho t}\log\left(\frac{\lambda(0)}{1+\lambda(0)}\right)dt \\
&\quad - \frac{1-e^{-\rho T}}{\rho}\log\left(\frac{\lambda(0)}{1+\lambda(0)}\right) = 0. \qquad \blacksquare
\end{aligned}$$

References

Allen, F., and D. Gale, 1994, "Limited Market Participation and Volatility of Asset Prices," *American Economic Review*, 84, 933–955.

Aiyagari, S. R., and M. Gertler, 1991, "Asset Returns with Transaction Costs and Uninsured Individual Risk," *Journal of Monetary Economics*, 27, 311–331.

Balasko, Y., D. Cass, and K. Shell, 1995, "Market Participation and Sunspot Equilibria," *Review of Economic Studies*, 62, 491–512.

Balasko, Y., D. Cass, and P. Siconolfi, 1990, "The Structure of Financial Equilibrium with Exogenous Yields. The Case of Restricted Participation," *Journal of Mathematical Economics*, 19, 195–216.

Basak, S., 1996a, "An Intertemporal Model of International Capital Market Segmentation," *Journal of Financial and Quantitative Analysis*, 31, 161–188.

Basak, S., 1996b, "A Model of Dynamic Equilibrium Asset Pricing with Extraneous Risk," working paper, University of Pennsylvania.

Black, F., 1974, "International Capital Market Equilibrium with Investment Barriers," *Journal of Financial Economics*, 1, 337–352.

Blume, M. E., and S. P. Zeldes, 1994, "Household Stockownership Patterns and Aggregate Asset Pricing Theories," working paper, University of Pennsylvania.

Breeden, D. A., 1979, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics*, 7, 265–296.

Constantinides, G. M., 1990, "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, 98, 519–543.

Constantinides, G. M., and D. Duffie, 1996, "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104, 219–240.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363–384.

Cuny, C. J., 1993, "The Role of Liquidity in Futures Markets Innovation," *Review of Financial Studies*, 6, 57–78.

Cuoco, D., and H. He, 1994a, "Dynamic Aggregation and Computation of Equilibria in Finite-Dimensional Economies with Incomplete Financial Markets," working paper, University of Pennsylvania.

Cuoco, D., and H. He, 1994b, "Dynamic Equilibrium in Infinite-Dimensional Economies with Incomplete Financial Markets," working paper, University of Pennsylvania.

Errunza, V., and E. Losq, 1985, "International Asset Pricing under Mild Segmentation: Theory and Test," *Journal of Finance*, 40, 105–124.

Errunza, V., and E. Losq, 1989, "Capital Flow Controls, International Asset Pricing, and Investors' Welfare: A Multi-Country Framework," *Journal of Finance*, 44, 897–914.

Eun, C. S., and S. Janakiramanan, 1986, "A Model of International Asset Pricing with a Constraint on the Foreign Equity Ownership," *Journal of Finance*, 41, 1025–1037.

He, H., and N. D. Pearson, 1991, "Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case," *Journal of Economic Theory*, 54, 259–304.

Heaton, J., and D. J. Lucas, 1996, "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," *Journal of Political Economy*, 104, 443–487.

Hirshleifer, D., 1988, "Residual Risk, Trading Costs, and Commodity Futures Risk Premia," *Review of Financial Studies*, 1, 173–193.

Huang, C.-f., 1987, "An Intertemporal General Equilibrium Asset Pricing Model: The Case of Diffusion Information," *Econometrica*, 55, 117–142.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve, 1990, "Existence and Uniqueness of Multi-Agent Equilibrium in a Stochastic, Dynamic Consumption/Investment Model," *Mathematics of Operations Research*, 15, 80–128.

Karatzas, I., J. P. Lehoczky, S. E. Shreve, and G.-l. Xu, 1991, "Martingale and Duality Methods for Utility Maximization in an Incomplete Market," *SIAM Journal of Control and Optimization*, 29, 702–730.

Karatzas, I., and S. E. Shreve, 1988, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

Kimball, M. S., 1990, "Precautionary Saving in the Small and in the Large," *Econometrica*, 58, 53–73.

Leland, H. E., 1968, "Saving and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics*, 82, 465–473.

Mankiw, N. G., and S. P. Zeldes, 1991, "The Consumption of Stockholders and Non-Stockholders," *Journal of Financial Economics*, 29, 97–112.

Mehra, R., and E. C. Prescott, 1985, "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–161.

Merton, R. C., 1987, "A Simple Model of Capital Market Equilibrium with Incomplete Information," *Journal of Finance*, 42, 483–510.

Protter, P., 1990, *Stochastic Integration and Differential Equations*, Springer-Verlag, New York.

Saito, M., 1996, "Limited Participation and Asset Pricing," working paper, University of British Columbia.

Sellin, P., and I. Werner, 1993, "International Barriers in General Equilibrium," *Journal of International Economics*, 34, 2107–2138.

Serrat, A., 1995, "An Equilibrium Analysis of Liquidity Constraints," working paper, Massachusetts Institute of Technology.

Stulz, R. M., 1981, "On the Effects of Barriers to International Investment," *Journal of Finance*, 36, 923–934.

Subrahmanyam, M. G., 1975, "On the Optimality of International Capital Market Integration," *Journal of Financial Economics*, 2, 3–28.

Weil, P., 1989, "The Equity Premium Puzzle and the Riskfree Rate Puzzle," *Journal of Monetary Economics*, 24, 401–421.